NOETHER SYMMETRIES OF BIANCHI TYPE II SPACETIME METRICS

BY

MAHMOOD RAJIH TARAYRAH

A Thesis Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

In

MATHEMATICS

MAY 2014

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **MAHMOOD RAJIH TARAYRAH** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

Thesis Committee

Prof. Ashfaque H. Bokhari (Adviser)

Prof. F.D.Zaman (Member)

Prof. Hassan Azad (Member)

Dr. Husain S. Al-Attas
Department Chairman

Dr. Salam A. Zummo
Dean of Graduate Studies

21/5/14
Date

To The Prophet ***MUHAMMAD -PBUH-***
To my country ***Palestine***
my parents
brothers and sisters
my friends
who supported me throughout my study

ACKNOWLEDGMENTS

All praise due to Almighty Allah, Most Gracious and Most Merciful, for his immense beneficence and blessings. He bestowed upon me health, knowledge and patience to complete this work. May peace and blessings be upon prophet Muhammad (PBUH), His Family, His Companions, and those who follow him.

Thereafter, acknowledgement is due to KFUPM for the support extended towards my research through its remarkable facilities and for granting me the opportunity to pursue graduate studies. I acknowledge, with deep gratitude and appreciation, the inspiration, encouragement, valuable time and continuous guidance given to me by my thesis advisor, Dr. Ashfaque Bokhari. I am highly grateful and thankful to my thesis committee, Dr. Hasan Azad and Dr. F.D.Zaman for their valuable guidance, suggestions and motivations.

My heartfelt thanks are due to my parents, brothers and sisters for their prayers, guidance, and moral support throughout my academic life. My parents' advice, to strive for excellence has made all this work possible. Thanks for all professors of Mathematics and Statistics department and whoever's contributed in this work by any means.

TABLE OF CONTENTS

LIST OF TABLES

THESIS ABSTRACT

NAME:	Mahmood Rajih Tarayrah
TITLE OF STUDY:	Noether Symmetries of Bianchi Type II Spacetime Metrics
MAJOR FIELD:	Mathematics
DATE OF DEGREE:	May, 2014

In this thesis a complete classification of Noether symmetries of Bianchi type II spacetime is obtained. In the process of this classification we find several differential constraints that are satisfied by the coefficients of unknown Bianchi type II metric coefficients. Keeping in view these differential constraints, we classify the complete Noether symmetry groups according to their dimensions and other algebraic properties. In particular, we construct the Lie algebras of these Noether symmetries in each case and identify type of their algebras. The motivation of this work comes from the fact that the considered spacetime models are of interest in general relativity and represent some interesting solutions of the Einstein field equations.

ملخص الرسالة

الاسم الكامل: محمود راجح طرايره

عنوان الرسالة: تماثلات نويثر لمقياس بيانكي الزمكان النوع الثاني

التخصص: الرياضيات

تاريخ الدرجة العلمية: أيار 2014

في هذه الاطروحة تم الحصول على تصنيف كامل من تماثلات نويثر لمقياس بيانكي الزمكان النوع الثاني. في عملية التصنيف نجد العديد من القيود التفاضلية والتي تتحقق عن طريق معاملات مقياس بيانكي غير المعروفة. واضعا بالاعتبار هذه المعاملات، قمنا بتصنيف مجموعات كاملة لتماثلات نويثر وفقا لأبعادها وخصائصها الجبرية. وبشكل خاص قمنا بإنشاء جبور لي المكونة من تماثلات نويثر في كل حالة وتحديد نوع هذه الجبور.

الدافع لهذا العمل يأتي من حقيقة أن مقياس بيانكي له أهمية كبيرة في النسبية العامة ويمثل بعض الحلول المهمة لمعادلات أينشتاين.

CHAPTER 1

INTRODUCTION

Solving differential equations has a great importance in applied mathematics. Many of non-linear differential equations do not have an exact solution, so we need to develop methods of finding solutions of these equations. In 1774, Euler and Lagrange became interested in finding a solution of differential equations given by the formula, $y = \int H(x)e^{\alpha x}dx \quad and \quad y = \int H(x)x^{\alpha}dx.$

In 1785, Laplace established a very clever technique to solve the differential equations, by transforming the original equations into a new equations which are easier to solve.

In 1881 Sophus Lie established a very important method to find the solutions of differential equations, these method known as Lie symmetries [1-5]. A symmetry of a system is a transformation whose action leaves the system invariant. So, if the one-parameter Lie groups of transformation leaves a differential equation unchanged, then this method can be used to reduce the order of the differential equation [6,7]. As of now, these techniques are being widely used in finding exact solutions of non-linear differential (both ordinary and partial) equations [6-12]. More recently, the Lie symmetry methods have been applied to study symmetry structures of non-linear differential equations that arise in curved 4-spacetime geometries [11, 12]. It is shown that due to the presence of curvature, the dimensions of the Lie symmetry groups of the differential equations decrease as opposed to the spaces with no curvature [13-15].

Another interesting symmetries which has a great importance in the field of applied mathematics is called Noether symmetry, named after Emmy Noether , born in 1882 in Erlangen in southern Germany [7]. "From 1889 to 1897 Emmy's name appears in the class lists of the Municipal School for Higher Education of Daughters in Erlangen, first as Emmy, later as Emma Noether. In 1900 Miss Noether, registered in Ansbach for the examination required by the state of Bavaria for teachers of French and English. Noether achieved an average of 1.2 in each language, for which she was awarded an overall grade of 'very good'. She decided to continue her studies at the university level. At that time in Germany, only very few women attended university lectures. Emmy Noether became one of them. After graduation Emmy spent her first semester at the university in Gottngen. Under Gordan's influence she wrote a paper on the theory of invariants, entitled, On the construction of the systems of forms for the ternary biquadratic form. It was registered as her doctoral dissertation in 1907"[29]. Noether's work is divided into three periods. In (1908:1919), she contributed in the theories of number fields, algebraic invariants and the calculus of variations, these significant contributions and Noether's theorem, received high recognition and had a great influence in the development of modern physics[30]. In (1920:1926), she began work on abstract algebra". In her paper (Theory of Ideals in Ring Domains, 1921), she developed theory in abstract algebra of commutative rings which had an important applications[31]. In (1927:1935), she worked on hyper complex numbers and noncommutative algebras and unified the theory of modules and ideals with the representation theory of groups . In addition to the previous works, Noether shared her ideas with other mathematicians, in fields other than her original work, such as algebraic topology[29].

The Noether symmetries are associated with differential equations that possess Lagrangians. For differential equations possessing Lagrangian, the amount of work involved is halved. This is because one uses the order of the Lagrangian in the prolongation process while the other uses the order of the differential equation. When applied to spacetime, as a bonus, the Lagrangian of the Euler Lagrange equations (geodesic equations) come 'naturally'. Moreover, the algebra of Noether symmetries generated by the Lagrangian contains the Killing vectors generated by the corresponding metric[16].

In recent years the Noether symmetries have been the interest of many researchers and nice results have been published. In 1918 Noether proposed her famous paper on Noether symmetry theory and the invariance of Hamilton actions under infinitesimal transformation [17]. Vujanovic and Djukic[18] studied non-conservative holonomic systems and they investigated the conserved quantities and the Noether symmetries of this system. The mechanical systems and their Noether symmetries are of great importance in mathematics and some physical principles [19]. There is a method for solving differential equations called Birkhoff-Noether symmetry method. The procedure in this method is writing the differential equations in terms of Birkhoff's equations, and then by finding the Noether symmetry for these systems we obtain first integrals for differential equations [20]. Also, the Noether symmetry approach is applied to study the Einstein equations minimally coupled with a scalar field, in some cases of Bianchi universe models [21]. In this paper[22], New forms relating the Euler-Lagrange, Noether and Lie-Backlund operators are obtained . Some theories in the Noether symmetry and gravity of Kantowski-Sachs models, Bianchi I and Bianchi III are considered [23]. To understand the comparison between Noether symmetries and conformal Killing vectors, Bokhari et. al, published results on Noether symmetries of Friedmann model [24]. In this work they proved that there are additional conservation laws are admitted by the flat Friedman model and these conserved quantities are not obtained from the Killing or the conformal Killing vectors. These conservation laws provide a technique which is used for reducing the geodesic equations via the corresponding Noether symmetries. Tsamparlis et. al, studied the dynamical system and showed that the Lie symmetries of the equations of motion are created from a Lie algebra of projective collineations while Noether symmetries are created from the homothetic algebra [25]. By utilizing the Noether symmetry approach in cosmology, Jamil et.al published their results by applying this symmetry to a flat

Friedmann-Robertson-Walker metric and finding the tachyon potential[26]. In a recent work, Noether symmetries of Petrove III and Papapetrou metrics are studied. Considering the invariance of the action integral under one parameter Lie group of transformations, a large class of Noether symmetries is found [27].

In this thesis we intend to classify Noether symmetries for Bianchi type II spacetimes according to certain differential constraints arising on the metric coefficients during the process of classification of Noether symmetries and then we will construct the lie algebras associated with these Noether symmetries.

The other chapters of this thesis contain the following:

Chapter 2 contains the basic definitions, theorems and illustrations of Lie groups and representations of Lie algebras. In Chapter 3, we provide some basic operators and the derivation of the Noether symmetry formula, as an example, we find the Noether symmetries for a given Lagrangian.

In Chapter 4, we find a complete classification of the Noether symmetries for a Lagrangian associated with Bianche type II metric, and then we classify the type of the Lie algebras of these symmetries.

Chapter 5 contains a conclusion and some recommendations for this work.

CHAPTER 2

LIE POINT SYMMETRIES

2.1 Groups

Definition 2.1 *A group G is a set of elements with a law of composition ϕ between elements satisfying the following properties :*

1- Closure property:

For any elements x,$y \in G$, $\phi(x, y) \in G$.

2-Associative property :

For any elements x,y,$z \in G$, $\phi(x, \phi(y, z)) = \phi(\phi(x, y), z)$

3- Identity element : there is a unique element e of G such that for any element $x \in G$: $\phi(x, e) = \phi(e, x) = x$

4-Inverse element :

For any element $x \in G$, there exist a unique inverse element x^{-1} in G such that $\phi(x, x^{-1}) = \phi(x^{-1}, x) = e$

Definition 2.2 *A group G is abelian if $\phi(x,y) = \phi(y,x)$ for all elements x,y in G.*

2.2 Group Of Transformations

Definition 2.3 *Let $x = (x_1, x_2, ..., x_n)$ lie in region $D \subset R^n$. The set of transformations*

$$\bar{x} = T(x;\varepsilon) \tag{2.1}$$

defined for each x in D and parameter ε in set $M \subset R$, with $\phi(\varepsilon,\delta)$ defining a law of composition of parameters ε and δ in M, forms a one-parameter group of transformations on D if the following conditions hold :

1- The transformations are one-to-one and onto D.

2- (M, ϕ) is a group.

3- $T(x, \varepsilon_0) = x$ for the identity element ε_0 and for each x in D.

4- If $\bar{x} = T(x,\varepsilon)$, $\bar{\bar{x}} = T(\bar{x},\delta)$, then $\bar{\bar{x}} = T(x, \phi(\varepsilon,\delta))$

2.3 One-Parameter Lie Group Of Transformations

Definition 2.4 *The one-parameter group of transformations $\bar{x} = T(x,\varepsilon)$ considered in definition (2.3) is a one-parameter Lie group of transformation if :*

1- M is an interval in R.

2- T is infinitely differentiable with respect to x in D and an analytic function of ε in M.

3- $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ.

2.3.1 Examples Of One-Parameter Lie Group Of Transformations

Example 2.1 *The group of translations*

Consider the group of translations

$$\begin{aligned} \bar{x} &= T_1(x; \varepsilon) = x + \varepsilon, \\ \bar{y} &= T_2(y; \varepsilon) = y, \quad \varepsilon \in R. \end{aligned} \tag{2.2}$$

Repeating the transformation,

$\bar{\bar{x}} = T_1(\bar{x}, \delta) = \bar{x} + \delta = x + \varepsilon + \delta,$

$\bar{\bar{y}} = T_2(\bar{y}, \delta) = \bar{y} = y.$

Here $\phi(\varepsilon, \delta) = \varepsilon + \delta$ and $\varepsilon = 0$ corresponds to the identity element.

Example 2.2 *Group of scalings*

$\bar{x} = ax,$

$\bar{y} = a^2 y, \ 0 < a < \infty$.

Here $\phi(a, b) = ab$, and the identity element corresponds to $a = 1$.

2.4 Infinitesimal Transformations

Let $\bar{x} = T(x;\varepsilon)$ be a one-parameter Lie group of transformations, we expand it about $\varepsilon = 0$ to get,

$$\begin{aligned}\bar{x} &= x + \varepsilon \left[\left. \tfrac{\partial T(x,\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} \right] + \tfrac{1}{2}\varepsilon^2 \left[\left. \tfrac{\partial^2 T(x,\varepsilon)}{\partial \varepsilon^2} \right|_{\varepsilon=0} \right] + \ldots \\ &= x + \varepsilon \left[\left. \tfrac{\partial T(x,\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} \right] + O(\varepsilon^2).\end{aligned}$$

Put $\xi(x) = \left[\left. \tfrac{\partial T(x,\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} \right]$, then the components of $\xi(x)$ are called the infinitesimals of Eq.(2.1).

Theorem 2.3 *(First Fundamental Theorem of Lie)*

Consider the Lie group of transformations given by the Eq.(2.1), then there exist a parametrization $\psi(\varepsilon)$ such that Eq.(2.1) is equivalent to the solution of an initial value problem for a system of first-order ODEs given by

$$\frac{d\bar{x}}{d\psi} = \xi(\bar{x}), \tag{2.3}$$

with $\bar{x} = x$ *when* $\psi = 0$.

In particular,

$$\psi(\varepsilon) = \int_0^{\varepsilon} \Gamma(\varepsilon')d\varepsilon', \tag{2.4}$$

$$\Gamma(\varepsilon) = \left. \frac{\partial \phi(a,b)}{\partial b} \right|_{(a,b)=(\varepsilon^{-1},\varepsilon)} \tag{2.5}$$

and $\Gamma(0) = 1$.

Example 2.4 *Consider the group of translation given by Eq.(2.2) with a law of composition given by $\phi(\alpha,\beta) = \alpha + \beta$, and $\varepsilon^{-1} = -\varepsilon$, then*

$\Gamma(\varepsilon) = \frac{\partial \varphi(\alpha,\beta)}{\partial \beta}\Big|_{(\alpha,\beta)=(-\varepsilon,\varepsilon)} = 1.$

Now, $\quad \xi(x) = \frac{\partial T(x,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = (1,0).$

As a result,

$$\frac{d\bar{x}}{d\varepsilon} = 1, \ \frac{d\bar{y}}{d\varepsilon} = 0, \tag{2.6}$$

with $\quad \bar{x} = x, \ \bar{y} = y, \ \varepsilon = 0.$ *Hence,the solution of the initial value problem (2.6) is given by Eq.(2.2).*

Definition 2.5 *Let* $\xi(x) = (\xi_1(x), \xi_2(x), ..., \xi_n(x))$ *be the infinitesimals of the Lie group of transformations (2.1), then the infinitesimal generator(operator) of (2.1) is given by:*

$$X = X(x) = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \tag{2.7}$$

where $x = (x_1, x_2, ..., x_n) \in R^n$.

Example 2.5 *For the scaling group :*

$$\bar{x} = ax, \bar{y} = a^2 y, \ \ 0 < a < \infty. \tag{2.8}$$

We have the infinitesimals $\xi(x,y) = \frac{\partial \bar{x}}{\partial a}\big|_{a=0} = x,$ *and* $\eta(x,y) = \frac{\partial \bar{y}}{\partial a}\big|_{a=0} = 0.$

Hence the infinitesimal generator is

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} = x\frac{\partial}{\partial x}. \tag{2.9}$$

Now the following theorem is useful in constructing a transformation group from its infinitesimals.

Theorem 2.6 *The one parameter Lie group of transformations (2.1) is equivalent to*

$$\bar{x} = e^{\varepsilon X} x = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} X^k x, \tag{2.10}$$

where X is the infinitesimal generator of (2.1).

Example 2.7 *Consider the infinitesimal generator*

$$X = y\frac{\partial}{\partial x} + -x\frac{\partial}{\partial y}, \tag{2.11}$$

we can determine the transformation group that corresponds to this generator using theorem (2.6) as follows:

$$\begin{aligned}\bar{x} = e^{\varepsilon X} x = \sum_{k=0}^{\infty} \tfrac{\varepsilon^k}{k!}\mathrm{X}^k x = &\left(1 - \tfrac{\varepsilon^2}{2!} + \tfrac{\varepsilon^4}{4!} + ...\right) x + \left(\varepsilon - \tfrac{\varepsilon^3}{3!} + \tfrac{\varepsilon^5}{5!} + ...\right) y \\ = &x\cos\varepsilon + y\sin\varepsilon.\end{aligned} \tag{2.12}$$

Similarly, $\bar{y} = e^{\varepsilon X} y = -x\sin\varepsilon + y\cos\varepsilon$, which is a rotation group in the plane.

2.5 Invariant Functions

Definition 2.6 *An infinitely differentiable function $G(x)$ is called an invariant function of the one-parameter Lie group of transformations*

$$\bar{x} = T(x;\varepsilon), \tag{2.13}$$

if and only if for any group of transformation given by Eq.(2.13),

$$G(\bar{x}) \equiv G(x). \tag{2.14}$$

Theorem 2.8 *We say that $G(x)$ is an invariant function under a Lie group of transformation if and only if,*

$$XG(x) \equiv 0. \tag{2.15}$$

2.6 Extended Transformations(Prolongations)

In order to find the infinitesimal transformation for a nth order differential equation we prolong the corresponding infinitesimal generator to the nth order.

2.6.1 Prolonged Transformations : One Dependent and One Independent Variable

In this section we study the invariance of a kth-order differential equation with independent variable x and dependent variable y, our goal is to find admitted one-parameter Lie group of transformations given by

$$\begin{aligned} \bar{x} &= X(x,y;\varepsilon), \\ \bar{y} &= Y(x,y;\varepsilon). \end{aligned} \tag{2.16}$$

We will extend transformations given by Eq.(2.16) to $(x, y_1, y_2, ..., y_k)$-space, where $y_k = y^{(k)} = \frac{d^k y}{dx^k}, k \geq 1$.

Theorem 2.9 *The first extension of (2.16) is the one-parameter Lie group of transformation acting on (x, y, y_1)-space given by*

$$\begin{aligned} \bar{x} &= X(x,y;\varepsilon), \\ \bar{y} &= Y(x,y;\varepsilon), \\ \bar{y}_1 &= Y_1(x,y,y_1;\varepsilon), \end{aligned} \tag{2.17}$$

where $\bar{y}_1$ is given by

$$\bar{y}_1 = \frac{\frac{\partial \bar{y}}{\partial x} + y_1 \frac{\partial \bar{y}}{\partial y}}{\frac{\partial \bar{x}}{\partial x} + y_1 \frac{\partial \bar{x}}{\partial y}}. \tag{2.18}$$

Theorem 2.10 *The kth extension of (2.16) is the following one-parameter Lie group of transformations acting on $(x, y_1, y_2, ..., y_k)$-space:*

$$\begin{aligned} \bar{x} &= X(x,y;\varepsilon), \\ \bar{y} &= Y(x,y;\varepsilon), \\ \bar{y}_1 &= Y_1(x,y,y_1;\varepsilon), \\ &\vdots \\ \bar{y}_k &= Y_k(x,y,y_1,\ldots,y_k) = \frac{\frac{\partial Y_{k-1}}{\partial x} + y_1 \frac{\partial Y_{k-1}}{\partial y} + \ldots + y_k \frac{\partial Y_{k-1}}{\partial y_{k-1}}}{\frac{\partial X}{\partial x} + y_1 \frac{\partial X}{\partial y}}. \end{aligned} \tag{2.19}$$

Definition 2.7 *The total derivative is given by*

$$D = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y} + y_2 \frac{\partial}{\partial y_1} + \ldots + y_{n+1} \frac{\partial}{\partial y_n} + \ldots \tag{2.20}$$

2.6.2 The Prolonged Infinitesimal Transformations:One Dependent and One Independent Variable

Consider the one-parameter Lie group of transformations:

$$\begin{aligned} \bar{x} &= x + \varepsilon \xi(x, y) + O(\varepsilon^2), \\ \bar{y} &= y + \varepsilon \eta(x, y) + O(\varepsilon^2), \end{aligned} \tag{2.21}$$

which has infinitesimals $\xi(x, y), \eta(x, y)$. The corresponding symmetry generator is written as:

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}. \tag{2.22}$$

The kth extension of Eq.(2.21) is:

$$\begin{aligned} \bar{x} &= x + \varepsilon \xi(x, y), \\ \bar{y} &= y + \varepsilon \eta(x, y), \\ \bar{y}_1 &= y_1 + \varepsilon \eta^1(x, y, y_1), \\ &\vdots \\ \bar{y}_k &= y_k + \varepsilon \eta^k(x, y, y_1, \ldots, y_k), \end{aligned} \tag{2.23}$$

where, $\xi(x,y), \eta(x,y), \eta^1(x,y,y_1), \ldots, \eta^k(x,y,y_1,\ldots,y_k)$ are the kth-extended infinitesimals. Accordingly the prolonged symmetry generator is given by

$$X^k = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^1(x,y,y_1)\frac{\partial}{\partial y_1} + \ldots + \eta^k(x,y,y_1,\ldots,y_k)\frac{\partial}{\partial y_k}. \quad (2.24)$$

The prolonged infinitesimals η^k, $k \geq 1$ are determined by the following theorem:

Theorem 2.11 *The kth-extended infinitesimals are determined by the following relation:*

$$\eta^k(x,y,y_1,\ldots,y_k) = D\eta^{k-1}(x,y,y_1,\ldots,y_{k-1}) - y_k D\xi(x,y), k \geq 1, \quad (2.25)$$

where $\eta^0 = \eta(x,y), \quad \eta^1 = \eta_x + y_1(\eta_y - \xi_x) - y_1^2\xi_y.$

2.6.3 Prolonged Transformations and Prolonged Infinitesimals:'m' Dependent and 'n' Independent Variables

In study the system of differential equations, the case of m dependent variables $v = (v^1, v^2, \ldots, v^m)$ and n independent variables $x = (x_1, x_2, \ldots, x_n)$, $v = v(x)$ arises. So we extend the transformations from $(x,v) - space$ to $(x, v, v^{(1)}, v^{(2)}, \ldots, v^{(k)}) - space$ where $v^{(k)}$ represents all kth derivatives of v with respect to x.

Consider the one-parameter Lie group of transformations given by,

$$\begin{aligned} \bar{x}_i &= X_i(x,v), i = 1,2,\ldots,n \\ \bar{v}^a &= V^a(x,v), a = 1,2,\ldots,m \end{aligned} \tag{2.26}$$

The corresponding infinitesimal generator is given by the formula,

$$X = \xi_i(x,v)\frac{\partial}{\partial x_i} + \eta^a(x,v)\frac{\partial}{\partial v^a}. \tag{2.27}$$

The kth extended Lie group of transformations of Eq.(2.26)is the following group of transformations acting on $(x, v, v^{(1)}, v^{(2)}, \ldots, v^{(k)}) - space$. Here $v^{(1)} = v_i^a, v^{(2)} = v_{ij}^a, \ldots, v^{(k)} = v_{i_1 i_2 \ldots i_k}^a$, where indices take the values $a = 1,2,\ldots,m$ and $i, j, i_1, \ldots, i_k = 1,2,\ldots,n$ is given by

$$\begin{aligned} &\bar{x}_i = x_i + \varepsilon\xi_i(x,v), \\ &\bar{v}^a = v^a + \varepsilon\eta^a(x,v), \\ &\bar{v}_i = v_i^a + \varepsilon\eta_i^{(1)a}(x,v,v^{(1)}), \\ &\vdots \\ &\bar{v}_{i_1 i_2 \ldots i_k}^a = v_{i_1 i_2 \ldots i_k}^a + \varepsilon\eta_{i_1 i_2 \ldots i_k}^{(k)a}(x,v,v^{(1)},v^{(2)},\ldots,v^{(k)}). \end{aligned} \tag{2.28}$$

The extended infinitesimals $\eta_i^{(1)a}, \eta_{i_1 i_2 \ldots i_k}^{(k)a}$ appearing in the above equation are determined by the following theorem:

Theorem 2.12 *The extended infinitesimals for Eq.(2.26) are given by the for-*

mula,

$$\eta_i^{(1)a} = D_i\eta^a - v_j^a D_i\xi_j$$
$$\eta_{i_1 i_2 \ldots i_k}^{(k)a} = D_{i_k}\eta_{i_1 i_2 \ldots i_{k-1}}^{(k-1)a} - v_{i_1 i_2 \ldots i_{k-1} j}^a D_{i_k}\xi_j \tag{2.29}$$

where $i_s = 1, 2, \ldots, n$ *for* $s = 1, 2, \ldots, k.k \geq 2$ *and* D_i *is the total differential operator given by,*

$$D_i = \frac{\partial}{\partial x_i} + v_i^a \frac{\partial}{\partial v^a} + v_{ij}^a \frac{\partial}{\partial v_j^a} + \ldots + v_{i i_1 i_2 \ldots i_n}^a \frac{\partial}{\partial v_{i_1 i_2 \ldots i_n}^a} + \ldots \tag{2.30}$$

2.7 Lie Algebras of Operators

To study the infinitesimal transformations, we introduce algebraic structures called Lie algebras. For this purpose we need the following definitions:

2.7.1 The Commutator

Definition 2.8 *Let any two symmetry generators be given by* X_i, X_j*. Then the commutator(Lie bracket)[,] of these two generators is given by the following formula:*

$$[X_i, X_j] = X_i X_j - X_j X_i \tag{2.31}$$

Theorem 2.13 *The commutator satisfies the following properties:*

1-(Skew symmetric property) Let X_i, X_j be any two symmetry operators then

$$[X_i, X_j] = -[X_j, X_i]. \tag{2.32}$$

We can conclude from this property that $[X_i, X_i] = 0$.

2-(Jacobi Identity)

$$[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0, \tag{2.33}$$

for any three symmetry operators X_i,X_j,X_k.

3-(Bilinear property)

$$[aX_i + bX_j, X_k] = a[X_i, X_k] + b[X_j, X_k], \tag{2.34}$$

$$[X_k, aX_i + bX_j] = a[X_k, X_i] + b[X_k, X_j], \tag{2.35}$$

where a,b are arbitrary constants.

2.7.2 The Table of Commutators

It is appropriate to sort the commutator in a table called the commutator table, where $[X_j, X_k]$ is the entry of the intersection of j row with k column. Using the property given by Eq.(2.32), the commutator table is represented by a skew-symmetric matrix with zeros on its diagonal.

Example 2.14 *Consider the symmetry generators*

$X_1 = e^y \frac{\partial}{\partial y}, X_2 = \frac{\partial}{\partial x}$ *and* $X_3 = 2x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, *then using (2.31) we find that:*

$[X_1, X_2] = 0, [X_1, X_3] = -X_1, [X_2, X_3] = 2X_2$.

In table form the above are written as

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$-X_1$
X_2	0	0	$2X_2$
X_3	X_1	$-2X_2$	0

Table 2.1: Commutator table for example (2.14)

2.7.3 Definition of Lie Algebra

A Lie algebra L is a vector space of symmetry generators together with the operation of Lie bracket $[,]$ satisfying the properties (2.32) and (2.33).

Example 2.15 *Let W be a finite dimensional vector space over R and let $L(W)$ be the set of all linear operators on W.Then $L(W)$ is a vector space over R.*

If we define the commutator by $[x_1, x_2] := x_1 \circ x_2 - x_2 \circ x_1$ for x_1,$x_2 \in L(W)$, where $\circ$ is the composition of maps.

Then $L(W)$ is a Lie algebra called the general linear algebra.

2.7.4 Adjoint Representation of Lie groups and Lie Algebras

Let L be a Lie group and let ℓ be its algebra. Define a map $\lambda : L \to Aut(L)$ by $\lambda(r) = \lambda_r$ and λ_r is given by $\lambda_r = rhr^{-1}$, for all h in L.

The derivative of λ_r at the identity is an automorphism of the Lie algebra ℓ.

$(d\lambda)_x : T_x\mathrm{L} \to T_{\lambda(x)}\mathrm{Aut}(\mathrm{L})$,

$(d\lambda)_e : T_e\mathrm{L} \to T_{\lambda(e)=e}\mathrm{Aut}(\mathrm{L})$,

$(d\lambda_r)_x : T_x\mathrm{L} \to T_{\lambda_r(x)}(\mathrm{L})$,

$(d\lambda_r)_e : T_e\mathrm{L} \to T_{\lambda_r(e)=e}(\mathrm{L})$,
where $T_e\mathrm{L}$ is the tangent space to the identity element e of L. We denote this map by Ad_r :

$(d\lambda_r)_e = Ad_r : \ell \to \ell$, such that Ad_r is a linear transformation of ℓ which preserves the commutator.

The map:

$$Ad : L \to Aut(\ell), \mathrm{r} \to Ad_r \tag{2.36}$$

is said to be the adjoint representation of L.

Now if we take the derivative of Eq.(2.36) we get the adjoint representation of the Lie algebra ℓ :

$$ad : \ell \to End(\ell), X \to ad_X.X \in \ell, \tag{2.37}$$

where

$$ad_X(Y) = [X, Y] \text{ for all } X, Y \in \ell. \tag{2.38}$$

To prove the relation (2.38) we proceed as follows

$$\begin{aligned}
ad_X(Y) &= d(Ad_X)_e(Y) \\
&= \lim_{\rho\to 0} \tfrac{(I+\rho X)Y(I+\rho X)^{-1}-Y}{\rho} \\
&= \lim_{\rho\to 0} \tfrac{(I+\rho X)Y(I-\rho X+(\rho X)^2+O(\rho^3))-Y}{\rho} \\
&= \lim_{\rho\to 0} \tfrac{(I+\rho X)YI-(I+\rho X)Y\rho X+(I+\rho X)Y(\rho X)^2+O(\rho^3))-Y}{\rho} \\
&= \lim_{\rho\to 0} \tfrac{(IYI+\rho XYI-IY\rho X+\rho XY\rho X+IY(\rho X)^2+\rho XY(\rho X)^2+O(\rho^3))-Y}{\rho} \\
&= \lim_{\rho\to 0} \tfrac{Y+\rho XY-YX\rho+XYX\rho^2+YX^2\rho^2+XYX^2\rho^2+O(\rho^3)-Y}{\rho} \\
&= \lim_{\rho\to 0} XY-YX+XYX\rho+YX^2\rho+XYX^2\rho+O(\rho^2) \\
&= XY-YX=[X,Y]\,.
\end{aligned}$$

Example 2.16 *The representation of Lie algebra is given by matrices.Consider the Lie algebra $M_{2\times 2}$: The set of all 2×2 matrices, which has standard basis $\beta=\{e_1,e_2,e_3,e_4\}$. Then:*

$$ad_{e_1}(e_1)=[e_1,e_1]=0,$$

$$ad_{e_1}(e_2)=e_1e_2-e_2e_1=e_2,$$

$$ad_{e_1}(e_3)=e_1e_3-e_3e_1=-e_3,$$

$$ad_{e_1}(e_4)=e_1e_4-e_4e_1=0.$$

Thus, the the adjoint representation of e_1 is the following matrix:

$$ad(e_1)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Similarly, we find that the adjoint representations of e_2, e_3, e_4 *are the following matrices:*

$$ad(e_2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad ad(e_3) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$ad(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.7.5 Structures of Lie Algebras

Definition 2.9 *The* killing form *of a Lie algebra* L *is the bilinear symmetric form* $\kappa : L \times L \to F$ *given by,*

$$\kappa(x, y) = tr(adx \circ ady), \;\; for\ all\ x, y \in L \tag{2.39}$$

Definition 2.10 *Let* L *be a Lie algebra, then its* center *is given by,*

$$Z(L) = \{x \in L : [x, y] = 0 \; for\, all\, y \in L\}. \tag{2.40}$$

L *is abelian if* $Z(L) = L$.

Definition 2.11 *The* radical *of a Lie algebra* L *is defined to be:*

$$\Re(L) = \{x \in L : \kappa(x, y) = 0 \ \ for\, all\, y \in L\}, \tag{2.41}$$

if $\Re(L) = 0$, *we say* L *is* semisimple.

Definition 2.12 *The derived series of* L *is the series with terms:*

$$\begin{aligned} &L^{(0)} = L \quad and \quad L^{(k)} = [L^{(k-1)}, L^{(k-1)}], k \geq 1. \\ &Then\ L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq ... \end{aligned} \tag{2.42}$$

Definition 2.13 *A Lie algebra* L *is* solvable *if* $L^{(k)} = 0$ *for some* k.

Example 2.17 *Let* L *be a two dimensional Lie algebra whose Lie bracket is given by* $[a, b] = b$, *then* L *is solvable since* $L^{(2)}$=$\{0\}$.

Theorem 2.18 *For any Lie algebra* L, *the following are equivalent:*

1) L *is semisimple.*

2) The killing form κ *of* L *is nondegenerate.*

3) $\Re(L) = \{0\}$.

Theorem 2.19 *If* L *is any Lie algebra then* $L/_{\Re(L)}$ *is* semisimple.

Example 2.20 *Let* $L = sl(2, R)$ *with basis:*

$$\ell = \left\{ x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

$[x, y] = h,\ [x, h] = -2x,\ [y, h] = 2y.$

The adjoint representations are given by :

$$ad(x) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\quad ad(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix},\quad ad(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then:

$\kappa(x, x) = 0, \kappa(x, y) = 4, \kappa(x, h) = 0$

$\kappa(y, y) = 0, \kappa(y, h) = 0, \kappa(h, h) = 8.$

The matrix of κ *is given by:*

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

This matrix is invertible which implies that κ *is nondegenerate and therefore* ℓ *is* semisimple. *The center and the radical of* ℓ *are given by:*

$Z(\ell) = \{x \in \ell : [x, y] = 0, for\, all\, y \in \ell\} = \{0\}.$

$\Re(\ell) = \{x \in \ell : \kappa(x, y) = 0, for\, all\, y \in \ell\} = \{0\}.$

CHAPTER 3

THE NOETHER SYMMETRIES

Besides to her significant development of abstract algebra (one of the major fields of mathematics), Noether proved a theorem which endures her name. This theorem says:'to each differentiable symmetry of the action of a physical system corresponds a conservation law.'Where the action of a physical system is the integral of a Lagrangian function over the time. Noether's theorem has become essential tool of physics and calculus of variations.

For example, for the time translation symmetry Noether's theorem shows that the energy of the system is conserved, translation in space symmetry gives conservation of momentum, while the rotation symmetry implies the conservation of angular momentum.

3.1 Basic Operators

Consider the independent variable

$x = (x_1, x_2, ..., x_n)$,

and the dependent variable

$v = (v^1, v^2, ..., v^m)$.

The derivatives of v with respect to x are given by

$$v_i^a = D_i(v^a), \quad v_{ij}^a = D_j D_i(v^a), \tag{3.1}$$

where

$$D_i = \frac{\partial}{\partial x_i} + v_i^a \frac{\partial}{\partial v^a} + v_{ij}^a \frac{\partial}{\partial v_j^a} + ..., \quad i = 1, 2, ..., n \tag{3.2}$$

is the total differentiation operator.

The system of partial differential equations for the n independent variables $x = (x_1, x_2, ..., x_n)$, and m dependent variables $v = (v^1, v^2, ..., v^m)$ is given by

$$E^\beta(x, v, v^{(1)}, ..., v^{(s)}) = 0, \quad \beta = 1, ..., m \tag{3.3}$$

where $v^{(1)}$ represents all first derivatives v_i^a and $v^{(s)}$, $s \geq 2$ represents all sth order derivatives of v with respect to x.

Definition 3.1 *(Euler-Lagrange Operator)*

We call

$$\frac{\delta}{\delta v^a} = \frac{\partial}{\partial v^a} + \sum_{r \geq 1} (-1)^r D_{i_1} ... D_{i_r} \frac{\partial}{\partial v^a_{i_1 ... i_r}}, \ \ a = 1, ..., m \tag{3.4}$$

the Euler-Lagrange operator.

Let $L = L(x, v, v^{(1)}, ..., v^{(s)})$ *be the Lagrangian of the system (3.3), then the Euler-Lagrange equations are given by*

$$\frac{\delta}{\delta v^a}(L) = 0 \tag{3.5}$$

Example 3.1 *(Simple harmonic oscillator)*

Consider a Lagrangian $L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{k}{2} x^2$ *for the oscillator.*

Then the Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial x} - D_t \frac{\partial L}{\partial \dot{x}} = 0, \tag{3.6}$$

which implies that $-kx - D_t(m\dot{x}) = 0$, *and therefore* $kx + m\ddot{x} = 0$,

which corresponds to the equation of motion ($F = ma$*.)*

Example 3.2 *Using the Euler-Lagrange equation, we find the real-valued function* g *defined on* $[a, b]$ *with* $g(a) = c$, $g(b) = d$, *whose graph is as short as possible, as follows:*

$\ell(g) = \int_a^b \sqrt{1 + (g'(x))^2} dx,$

here L *is given by*

$L(x, y, y') = \sqrt{1 + (y')^2}$, *then*

$\frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}, \quad \frac{\partial L}{\partial y} = 0.$

By substituting in the Euler-Lagrange equation we get:

$\frac{\partial}{\partial x}\frac{g'(x)}{\sqrt{1+(g'(x))^2}} = 0,$ *which suggests that* $\frac{g'(x)}{\sqrt{1+(g'(x))^2}} = k = \text{constant},$ *and therefore* $g'(x) = \frac{k}{\sqrt{1-k^2}} := A,$ *which implies that* $g(x) = Ax + B,$ *which is a straight line.*

Definition 3.2 *The Lie-Backlund operator is given by*

$$X = \xi_i \frac{\partial}{\partial x_i} + \eta^a \frac{\partial}{\partial v^a} + \zeta_i^a \frac{\partial}{\partial v_i^a} + \zeta_{i_1 i_2}^a \frac{\partial}{\partial v_{i_1 i_2}^a} + ..., \tag{3.7}$$

where

$$\begin{aligned} \zeta_i^a &= D_i(\eta^a - \xi_j v_j^a) + \xi_j v_{ij}^a, \\ \zeta_{i_1 i_2}^a &= D_{i_1} D_{i_2}(\eta^a - \xi_j v_j^a) + \xi_j v_{j i_1 i_2}^a, \\ &\vdots \end{aligned} \tag{3.8}$$

Definition 3.3 *The differential function is an analytic function* $h(x, v, v^{(1)}, ..., v^{(s)})$ *of a finite number of variables.*

The vector space of all differential functions is denoted by $\Im$.

Theorem 3.3 *Two Lie-Backlund operators* X *and* $\tilde{X}$ *are equivalent if* $X - \tilde{X} = h^i D_i, \; h^i \in \Im$.

Definition 3.4 *We call the operator*

$$N^i = \xi_i + W^a \frac{\delta}{\delta v^a} + \sum_{k \geq 1} D_{i_1} ... D_{i_k}(W^a) \frac{\delta}{\delta v_{i i_1 ... i_k}^a}, \; i = 1, ..., n. \; a = 1, ..., m. \tag{3.9}$$

the Noether's operator associated with the Lie-Backlund operator, where $W^a = \eta^a - \xi_j v_j^a$ *is the Lie characteristic equations.*

Theorem 3.4 *The Noether, Lie-Backlund and Euler-Lagrange operators satisfy the following equation*

$$X + D_i\xi_i = W^a \frac{\delta}{\delta v^a} + D_i N^i. \;\; i = 1, ..., n. \;\; a = 1, ..., m. \tag{3.10}$$

Definition 3.5 *(Conservation laws)*

The tuple $T = (T^1, ..., T^n)$ *is called a conserved vector of Eq.(3.3) if it satisfies the property:*

$$D_j T^j = 0, \quad j = 1, 2, ..., n, \tag{3.11}$$

for all solutions of Eq.(3.3), and then Eq.(3.11) is called a conservation law for Eq.(3.3).

3.2 Derivation of Noether Symmetry

Given a Lagrangian $L = L(x^k, \dot{x}^k, t)$, the symmetry generator associated with this Lagrangian is given by $X = \xi\frac{\partial}{\partial t} + \eta^\alpha\frac{\partial}{\partial x^\alpha} + \dot{\eta}^\alpha\frac{\partial}{\partial \dot{x}^\alpha}$.

Consider a Lie point transformation

$$T_\varepsilon : \tilde{x}^k \to x^k + \varepsilon\xi^k(x^k), \tag{3.12}$$

with the corresponding symmetry generator given by

$$X = \xi^1(x^1, ..., x^k)\frac{\partial}{\partial x^1} + ... + \xi^k(x^1, ..., x^k)\frac{\partial}{\partial x^k}. \tag{3.13}$$

The Noether symmetry is invariant under the action integral,

$$W=\int\limits_{t_1}^{t_2} L(x^k,\dot{x}^k,t)dt \tag{3.14}$$

up to a gauge term $f(x^1,...,x^k)$,i.e.

$$T_\varepsilon:\int\limits_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{x}^k,\dot{\tilde{x}}^k,\tilde{t})d\tilde{t}=\int\limits_{t_1}^{t_2} L(x^k,\dot{x}^k,t)dt. \tag{3.15}$$

The action integral is invariant up to a gauge term if,

$$\int\limits_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{x}^k,\dot{\tilde{x}}^k,\tilde{t})d\tilde{t}=\int\limits_{t_1}^{t_2} L(x^k,\dot{x}^k,t)dt+\varepsilon f(x^1,...,x^k). \tag{3.16}$$

Equation (3.16) can be written as follows

$$\int\limits_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{x}^k,\dot{\tilde{x}}^k,\tilde{t})d\tilde{t}=\int\limits_{t_1}^{t_2} L(x^k,\dot{x}^k,t)dt+\varepsilon\int\limits_{t_1}^{t_2}\frac{d}{dt}f(x^1,...,x^k)dt. \tag{3.17}$$

Differentiating (3.17) with respect to $\tilde{t}$ gives,

$$\frac{d}{d\tilde{t}}\int\limits_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{x}^k,\dot{\tilde{x}}^k,\tilde{t})d\tilde{t}=\frac{d}{d\tilde{t}}\int\limits_{t_1}^{t_2} L(x^k,\dot{x}^k,t)dt+\varepsilon\frac{d}{d\tilde{t}}\int\limits_{t_1}^{t_2}\frac{d}{dt}f(x^1,...,x^k)dt, \tag{3.18}$$

Implying that

$$L(\tilde{x}^k,\dot{\tilde{x}}^k,\tilde{t})=\frac{dt}{d\tilde{t}}\frac{d}{dt}\int L(x^k,\dot{x}^k,t)dt+\varepsilon\frac{dt}{d\tilde{t}}\frac{d}{dt}\int\frac{d}{dt}f(x^1,...,x^k)dt. \tag{3.19}$$

If we put the lower limit to be zero,and the upper limit any value, Eq.(3.19) becomes:

$$\frac{d\tilde{t}}{dt}L(\tilde{x}^k,\dot{\tilde{x}}^k,\tilde{t}) = L(x^k,\dot{x}^k,t) + \varepsilon\frac{d}{dt}f(x^1,...,x^k) \tag{3.20}$$

From the transformation given by Eq.(3.12) we have:

$$\begin{aligned} \tilde{t} \quad &= t + \varepsilon\xi, \\ \tilde{x}^k \; &= x^k + \varepsilon\eta^k, \\ \dot{\tilde{x}}^k &= \dot{x}^k + \varepsilon\dot{\eta}^k, \\ \tfrac{d\tilde{t}}{dt} &= 1 + \varepsilon\tfrac{d\xi}{dt}. \end{aligned} \tag{3.21}$$

Substituting Eq.(3.20) into Eq.(3.21) gives,

$$(1+\varepsilon\frac{d\xi}{dt})[L(x^k+\varepsilon\eta^k,\dot{x}^k+\varepsilon\dot{\eta}^k,t+\varepsilon\xi) = L(x^k,\dot{x}^k,t) + \varepsilon\frac{d}{dt}f(x^1,...,x^k). \tag{3.22}$$

Rewriting above equation,

$$(1+\varepsilon\frac{d\xi}{dt})[L(x^k,\dot{x}^k,t)+\varepsilon\eta^k\frac{dL}{dx^k}+\varepsilon\dot{\eta}^k\frac{dL}{d\dot{x}^k}+\varepsilon\xi\frac{dL}{dt}] = L(x^k,\dot{x}^k,t)+\varepsilon\frac{d}{dt}f(x^1,...,x^k), \tag{3.23}$$

And simplifying we get

$$(\xi\frac{d}{dt}+\eta^k\frac{d}{dx^k}+\dot{\eta}^k\frac{d}{d\dot{x}^k})L + L(x^k,\dot{x}^k,t)\frac{d\xi}{dt} = \frac{df}{dt}. \tag{3.24}$$

The above equation can be written in an convenient form as:

$$XL + LD_t\xi = D_t f. \tag{3.25}$$

The above equation is called the Noether symmetry formula, and the Noether symmetry generator for a given Lagrangian ,X, is the one that satisfies the formula given by Eq.(3.25).

Definition 3.6 *A Lie-Backlund operator X is called a Noether symmetry if it satisfies the Noether symmetry formula,*

$$XL + LD_i\xi_i = D_i f_i, \quad f_i \in \Im, \ i = 1, 2, ..., n. \tag{3.26}$$

Theorem 3.5 *If $X, \tilde{X}$ are two equivalent Lie-Backlund operators. Then X is a Noether symmetry if and only if $\tilde{X}$ is.*

Corollary 3.6 *Any Noether symmetry is equivalent to a strict Noether symmetry.*

3.3 Example on Noether Symmetries for a Given Lagrangian

Consider the Lagrangian

$$L = \frac{1}{2}y'^2 - \frac{1}{2}y^2. \tag{3.27}$$

Using the Euler-Lagrange equation, we can find the differential equation admitted by this Lageangian

$$\frac{dL}{dy} = \frac{d}{dx}\frac{dL}{dy'}. \tag{3.28}$$

By substituting the above Lagrangian in Eq.(3.28) we get

$$\frac{d}{dy}(\frac{1}{2}y'^2 - \frac{1}{2}y^2) = \frac{d}{dx}\frac{d}{dy'}(\frac{1}{2}y'^2 - \frac{1}{2}y^2), \tag{3.29}$$

which yields the differential equation,

$$y'' + y = 0. \tag{3.30}$$

The Noether symmetry formula is given by

$$XL + LD_x\xi = D_x f, \tag{3.31}$$

and the symmetry generator X is given by

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial y'}, \tag{3.32}$$

where η^1 takes the form:

$$\eta^1 = \eta_x + y'\eta_y - y'\xi_x - y'^2\xi_y \tag{3.33}$$

and

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}. \tag{3.34}$$

Computing each of the term in Eq.(3.31), we obtain

$$y'\eta_x - y\eta + y'^2(\eta_y - \xi_x - \xi_y) + (\frac{1}{2}y'^2 - \frac{1}{2}y^2)(\xi_x + y'\xi_y) = f_x + y'f_y, \tag{3.35}$$

by comparing the coefficient of the derivatives of y and the constant term of the above equation yields the following system of differential equations:

$$y'^3: \qquad \xi_y = 0, \tag{3.36}$$

$$y'^2: \qquad \eta_y - \frac{1}{2}\xi_x = 0, \tag{3.37}$$

$$y': \qquad \eta_x - \frac{1}{2}y^2\xi_y - f_y = 0, \tag{3.38}$$

$$1: \qquad -\frac{1}{2}y^2\xi_x - y\eta - f_x = 0. \tag{3.39}$$

We conclude from Eq.(3.36) that:

$$\xi = R_1(x), \tag{3.40}$$

substituting Eq.(3.40) in Eq.(3.37) and integrating over y gives,

$$\eta = \frac{1}{2}R_1(x)_x y + R_2(x). \tag{3.41}$$

Substituting the value of η in Eq.(3.38) gives,

$$\frac{1}{2}R_1(x)_{xx}y + R_2(x)_x - f_y = 0, \tag{3.42}$$

integrate Eq.(3.42) over y we obtain

$$f(x,y) = \frac{1}{4}R_1(x)_{xx}y^2 + R_2(x)_x y + R_3(x). \tag{3.43}$$

Use Eqs.(3.40), (3.41) and (3.43) in Eq.(3.39), we get

$$(\frac{1}{4}R_1(x)_{xxx} + R_1(x)_x)y^2 + (R_2(x)_x x + R_2(x))y + R_3(x)_x = 0, \tag{3.44}$$

from the above equation we obtain the following differential equations:

$$\begin{aligned} &\tfrac{1}{4}R_1(x)_{xxx} + R_1(x)_x = 0, \\ &R_2(x)_x x + R_2(x) = 0, \\ &R_3(x)_x = 0. \end{aligned} \tag{3.45}$$

Solving the above equations leads to

$$\begin{aligned} R_1(x) &= c_1 + c_2\cos 2x + c_3\sin 2x, \\ R_2(x) &= c_4\cos x + c_5\sin x, \\ R_3(x) &= c_6. \quad c_i \in R,\ i = 1, ..., 6. \end{aligned} \tag{3.46}$$

Now, values of ξ, η and f are

$$\begin{aligned}
&\xi = c_1 + c_2\cos 2x + c_3\sin 2x,\\
&\eta = (-c_2\sin 2x + c_3\cos 2x)y + c_4\cos x + c_5\sin x, \qquad (3.47)\\
&f(x,y) = -(c_2\cos 2x + c_3\sin 2x)y^2 + (-c_4\sin x + c_5\cos x)y + c_6.
\end{aligned}$$

The obtained Noether symmetries and corresponding gauge terms are given by:

$$\begin{aligned}
&X_1 = \tfrac{\partial}{\partial x}, && f(x,y) = 0.\\
&X_2 = \cos 2x\tfrac{\partial}{\partial x} - y\sin 2x\tfrac{\partial}{\partial y}, && f(x,y) = -y^2\cos 2x.\\
&X_3 = \sin 2x\tfrac{\partial}{\partial x} + y\cos 2x\tfrac{\partial}{\partial y}, && f(x,y) = -y^2\sin 2x. \qquad (3.48)\\
&X_4 = \cos x\tfrac{\partial}{\partial y}, && f(x,y) = -y\sin x.\\
&X_5 = \sin x\tfrac{\partial}{\partial y}, && f(x,y) = y\cos x.
\end{aligned}$$

3.4 The Metric

Given a two dimensional manifold S, a metric g on S is a function which assigns to each point $p \in S$ a symmetric bilinear positive definite form $g_p : T_p \times T_p \to \mathbb{R}$ and which varies differentially with the points p.

In $\mathbb{R}^3$ we have the standard coordinates (x, y, z), and dx, dy, dz are linear forms means $\{dx_p, dy_p, dz_p\}$ is a basis of linear functions on $T_p(\mathbb{R}^3)$.

Therefore any metric on $\mathbb{R}^3$ is of the form:

$g_p = \sum\limits_{1\le i,j\le 3} g_{ij}(p)dx^i \otimes dx^j$, if g_p is symmetric then $g_{ij} = g_{ji}$.

To say that g varies differentially with p means that the functions g_{ij} are differentiable.

Example 3.7 *Standard metric on* $(\mathbb{R}^3)$ *is given by*

$$g = (dx)^2 + (dy)^2 + (dz)^2. \tag{3.49}$$

The induced metric on the plane $x + y + z = 1$ *is given by*

$$\begin{aligned}(dx)^2 + (dy)^2 + (d(1-x-y))^2 &= (dx)^2 + (dy)^2 + (-dx-dy)^2 \\ &= (dx)^2 + (dy)^2 + (dx)^2 + (dy)^2 + 2dxdy \\ &= 2(dx)^2 + 2(dy)^2 + 2dxdy.\end{aligned}$$

CHAPTER 4

NOETHER SYMMETRIES ASSOCIATED WITH BIANCHI TYPE II METRIC

4.1 Classification of The Noether Symmetries and their Adjoint Representations

In this section we classify Noether symmetries for Bianchi type II spacetimes according to the unknown functions involved in the metric representing these spacetimes. Bianchi type spacetimes play an important role in understanding and describing the early stages of evolution of the universe. In particular, the study of Bianchi type II is important because it represents familiar Friedmann type exact solution of the Einstein field equations. Our purpose here is not to discuss the relativistic properties of the model but to focus on classifying the Noether symmetries of such spacetimes for the unknown functions of time. Mathematically, these models are represented by a spacetime metric:

$$ds^2 = -dt^2 + A(t)^2 dx^2 + B(t)^2 (dy^2 + x^2 dz^2 - 2x dy dz) + C(t)^2 dz^2. \tag{4.1}$$

The Lagrangian associated with this metric is given by,

$$L = -\dot{t}^2 + A(t)^2 \dot{x}^2 + B(t)^2 (\dot{y}^2 + x^2 \dot{z}^2 - 2x\dot{y}\dot{z}) + C(t)^2 \dot{z}^2, \tag{4.2}$$

where the dot $(.)$ represents derivative with respect to the parameter $'s'$.

Since we are dealing with 4-spacetimes geometry, we write the form of the Noether vector field as

$$X = \mu\frac{\partial}{\partial s} + \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \varphi\frac{\partial}{\partial z} + \tau^1\frac{\partial}{\partial \dot{t}} + \xi^1\frac{\partial}{\partial \dot{x}} + \eta^1\frac{\partial}{\partial \dot{y}} + \varphi^1\frac{\partial}{\partial \dot{z}}, \tag{4.3}$$

where $\mu,\ \tau,\ \xi,\ \eta,\ \varphi$ are functions of $s,\, t,\, x,\, y,\, z.$ In this equation the $\tau^1,\, \xi^1,\, \eta^1,\, \varphi^1$ are the prolongations given by the formula:

$$\begin{aligned}
\tau^1 &= D_s\tau - \dot{t}D_s\mu,\\
\xi^1 &= D_s\xi - \dot{x}D_s\mu,\\
\eta^1 &= D_s\eta - \dot{y}D_s\mu,\\
\varphi^1 &= D_s\varphi - \dot{z}D_s\mu,
\end{aligned} \tag{4.4}$$

with

$$D_s = \frac{\partial}{\partial s} + \dot{t}\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} \tag{4.5}$$

is the total differential operator.

Applying the Noether symmetry formula given by

$$XL + LD_s\mu = D_s f, \tag{4.6}$$

on the Lagrangian given by (4.2), we obtain

$$[\mu\tfrac{\partial}{\partial s} + \tau\tfrac{\partial}{\partial t} + \xi\tfrac{\partial}{\partial x} + \eta\tfrac{\partial}{\partial y} + \varphi\tfrac{\partial}{\partial z} + \tau^1\tfrac{\partial}{\partial \dot{t}} + \xi^1\tfrac{\partial}{\partial \dot{x}} + \eta^1\tfrac{\partial}{\partial \dot{y}} + \varphi^1\tfrac{\partial}{\partial \dot{z}}][-\dot{t}^2 + A(t)^2\dot{x}^2 +$$
$$B(t)^2(\dot{y}^2 + x^2\dot{z}^2 - 2x\dot{y}\dot{z}) + C(t)^2\dot{z}^2] + [-\dot{t}^2 + A(t)^2\dot{x}^2 + B(t)^2(\dot{y}^2 + x^2\dot{z}^2 - 2x\dot{y}\dot{z}) +$$
$$C(t)^2\dot{z}^2][\tfrac{\partial}{\partial s} + \dot{t}\tfrac{\partial}{\partial t} + \dot{x}\tfrac{\partial}{\partial x} + \dot{y}\tfrac{\partial}{\partial y} + \dot{z}\tfrac{\partial}{\partial z}]\mu = [\tfrac{\partial}{\partial s} + \dot{t}\tfrac{\partial}{\partial t} + \dot{x}\tfrac{\partial}{\partial x} + \dot{y}\tfrac{\partial}{\partial y} + \dot{z}\tfrac{\partial}{\partial z}]f \tag{4.7}$$

Treating above equation as a polynomial in derivatives of (t, x, y, z), we obtain an over determined system of differential equations, given by:

$\dot{t}^3$: $$\mu_t = 0, \tag{4.8}$$

$\dot{x}^3$: $$A^2\mu_x = 0, \tag{4.9}$$

$\dot{y}^3$: $$B^2\mu_y = 0, \tag{4.10}$$

$\dot{z}^3$: $$(C^2 + B^2x^2)\mu_z = 0, \tag{4.11}$$

$\dot{t}^2$: $$\mu_s - 2\tau_t = 0, \tag{4.12}$$

$\dot{x}^2$: $$2AA'\tau + 2A^2\xi_x - A^2\mu_s = 0, \tag{4.13}$$

$$\dot{y}^2: \qquad 2BB'\tau + 2B^2\eta_y - B^2\mu_s - 2B^2x\varphi_y = 0, \tag{4.14}$$

$$\dot{z}^2: 2B^2x\xi + 2BB'x^2\tau + 2CC'\tau - 2B^2x\eta_z - C^2\mu_s - B^2x^2\mu_s + 2C^2\varphi_z + 2B^2x^2\varphi_z = 0, \tag{4.15}$$

$$\dot{t}\dot{x}: \qquad A^2\xi_t - \tau_x = 0, \tag{4.16}$$

$$\dot{t}\dot{y}: \qquad B^2\eta_t - \tau_y - B^2x\varphi_t = 0, \tag{4.17}$$

$$\dot{t}\dot{z}: \qquad -B^2x\eta_t - \tau_z + C^2\varphi_t + B^2x^2\varphi_t = 0, \tag{4.18}$$

$$\dot{x}\dot{y}: \qquad B^2\eta_x + A^2\xi_y - B^2x\varphi_x = 0, \tag{4.19}$$

$$\dot{x}\dot{z}: \qquad -B^2x\eta_x + A^2\xi_z + C^2\varphi_x + B^2x^2\varphi_x = 0, \tag{4.20}$$

$$\dot{y}\dot{z}: -B^2\xi - 2BB'x\tau - B^2x\eta_y + B^2\eta_z + B^2x\mu_s + C^2\varphi_y + B^2x^2\varphi_y - B^2x\varphi_z = 0, \tag{4.21}$$

$$\dot{t}: \qquad f_t + 2\tau_s = 0, \tag{4.22}$$

$$\dot{x}: \qquad -f_x + 2A^2\xi_s = 0, \tag{4.23}$$

$$\dot{y}: \qquad -f_y + 2B^2\eta_s - 2B^2x\varphi_s = 0, \tag{4.24}$$

$$\dot{z}: \qquad -f_z - 2B^2x\eta_s + 2C^2\varphi_s + 2B^2x^2\varphi_s = 0, \tag{4.25}$$

$$1: \qquad f_s = 0. \tag{4.26}$$

From Eqs.(4.8-4.11) it is clear that $\mu = \mu(s)$, whereas from Eq.(4.26) we find that $f = f(t, x, y, z)$.

In order to determine the Noether symmetries, we now differentiate Eq.(4.22) with respect to $'s'$ we get $\tau_{ss} = 0$, and therefore

$$\tau = r_1(t, x, y, z) + r_2(t, x, y, z)s. \tag{4.27}$$

Now differentiate Eq.(4.12) with respect to $'s'$ twice we get $\mu_{sss} = 0$, and since μ is a function of $'s'$ only, it takes the following form:

$$\mu = \frac{\alpha_1}{2}s^2 + \alpha_2 s + \alpha_3, \tag{4.28}$$

where α_1, α_2, α_3 are constants. Now differentiate Eq.(4.23) with respect to $'s'$ we obtain $\xi_{ss} = 0$, which suggests that ξ takes the form:

$$\xi = r_3(t, x, y, z) + r_4(t, x, y, z)s. \tag{4.29}$$

Now, differentiating Eqs.(4.24) and (4.25) with respect to $'s'$ we get $\eta_{ss} = 0$ and $\varphi_{ss} = 0$. Therefore, η and φ have the following form:

$$\eta = r_5(t, x, y, z) + r_6(t, x, y, z)s, \tag{4.30}$$

$$\varphi = r_7(t, x, y, z) + r_8(t, x, y, z)s. \tag{4.31}$$

Using the above values of μ and τ in Eq.(4.12) we get the following equation:

$$\alpha_2 + \alpha_1 s = \frac{\partial r_1}{\partial t} + \frac{\partial r_2}{\partial t}s, \tag{4.32}$$

by comparing the coefficients of $'s'$ and then integrating over $'t'$ we obtain

$$
\begin{aligned}
r_2 &= \tfrac{\alpha_1}{2}t + r_9(x,y,z),\\
r_1 &= \tfrac{\alpha_2}{2}t + r_{10}(x,y,z).
\end{aligned}
\tag{4.33}
$$

In the light of the above equations τ becomes:

$$\tau = \frac{\alpha_2}{2}t + r_{10}(x,y,z) + (\frac{\alpha_1}{2}t + r_9(x,y,z))s. \tag{4.34}$$

Substitute the values of τ and ξ in Eq.(4.16) leads to

$$A^2(\frac{\partial r_3}{\partial t} + \frac{\partial r_4}{\partial t}s) = \frac{\partial r_{10}}{\partial x} + \frac{\partial r_9}{\partial x}s. \tag{4.35}$$

Integrating the above equation over $'t'$ and comparing the coefficients of $'s'$ yields,

$$
\begin{aligned}
r_3 &= (\textstyle\int \frac{1}{A^2}dt)\frac{\partial r_{10}}{\partial x} + r_{11}(x,y,z),\\
r_4 &= (\textstyle\int \frac{1}{A^2}dt)\frac{\partial r_9}{\partial x}.
\end{aligned}
\tag{4.36}
$$

Using the above results in Eq.(4.29) we obtain,

$$\xi = (\int \frac{1}{A^2}dt)\frac{\partial r_{10}}{\partial x} + r_{11}(x,y,z) + (\int \frac{1}{A^2}dt)\frac{\partial r_9}{\partial x}s. \tag{4.37}$$

Using above results in Eq.(4.22) and integrating f with respect to time, we obtain

$$f = -\frac{\alpha_1}{2}t^2 - 2r_9(x,y,z)t + r_{12}(x,y,z). \tag{4.38}$$

At this stage we substitute the values of τ, η and φ in Eq.(4.17) and then comparing the coefficients of $'s'$ and integrating the resulting equation over $'t'$, we arrive at following expressions:

$$r_5 - xr_7 = \int \frac{1}{B^2}dt\,\frac{\partial r_{10}}{\partial y} + r_{13}(x,y,z), \tag{4.39}$$

$$r_6 - xr_8 = \int \frac{1}{B^2}dt\,\frac{\partial r_9}{\partial y}. \tag{4.40}$$

Now, using the values of ξ, η and φ in Eq.(4.19) gives,

$$\frac{\partial r_5}{\partial x} - x\frac{\partial r_7}{\partial x} = -\frac{A^2}{B^2}\int \frac{1}{A^2}dt\frac{\partial^2 r_{10}}{\partial x \partial y} - \frac{A^2}{B^2}\frac{\partial r_{11}}{\partial y}, \tag{4.41}$$

$$\frac{\partial r_6}{\partial x} - x\frac{\partial r_8}{\partial x} = -\frac{A^2}{B^2}\int \frac{1}{A^2}dt\frac{\partial^2 r_9}{\partial x \partial y}. \tag{4.42}$$

At this stage we use Eqs.(4.39) and (4.40) into Eqs.(4.41) and (4.43) respectively, and get

$$\begin{aligned} r_5 &= (-\tfrac{A^2}{B^2}\textstyle\int \tfrac{1}{A^2}dt - \int \tfrac{1}{B^2}dt)x\tfrac{\partial^2 r_{10}}{\partial x \partial y} + \int \tfrac{1}{B^2}dt\tfrac{\partial r_{10}}{\partial y} \\ &\quad -\tfrac{A^2}{B^2}x\tfrac{\partial r_{11}}{\partial y} + r_{13}(x,y,z) - x\tfrac{\partial r_{13}}{\partial x}, \\ r_6 &= (-\tfrac{A^2}{B^2}\textstyle\int \tfrac{1}{A^2}dt - \int \tfrac{1}{B^2}dt)x\tfrac{\partial^2 r_9}{\partial x \partial y} + \int \tfrac{1}{B^2}dt\tfrac{\partial r_9}{\partial y}, \\ r_7 &= (-\tfrac{A^2}{B^2}\textstyle\int \tfrac{1}{A^2}dt - \int \tfrac{1}{B^2}dt)\tfrac{\partial^2 r_{10}}{\partial x \partial y} - \tfrac{A^2}{B^2}\tfrac{\partial r_{11}}{\partial y} - \tfrac{\partial r_{13}}{\partial x}, \\ r_8 &= (-\tfrac{A^2}{B^2}\textstyle\int \tfrac{1}{A^2}dt - \int \tfrac{1}{B^2}dt)\tfrac{\partial^2 r_9}{\partial x \partial y}. \end{aligned} \tag{4.43}$$

Using above results into Eqs.(4.30) and (4.31), η and φ take the form:

$$\eta = (-\frac{A^2}{B^2}\int \frac{1}{A^2}dt - \int \frac{1}{B^2}dt)x\frac{\partial^2 r_{10}}{\partial x\partial y} + \int \frac{1}{B^2}dt\frac{\partial r_{10}}{\partial y} - \frac{A^2}{B^2}x\frac{\partial r_{11}}{\partial y} + r_{13}(x,y,z) - x\frac{\partial r_{13}}{\partial x}$$
$$+ [(-\frac{A^2}{B^2}\int \frac{1}{A^2}dt - \int \frac{1}{B^2}dt)x\frac{\partial^2 r_9}{\partial x\partial y} + \int \frac{1}{B^2}dt\frac{\partial r_9}{\partial y}]s, \tag{4.44}$$

$$\varphi = (-\frac{A^2}{B^2}\int \frac{1}{A^2}dt - \int \frac{1}{B^2}dt)\frac{\partial^2 r_{10}}{\partial x\partial y} - \frac{A^2}{B^2}\frac{\partial r_{11}}{\partial y} + [(-\frac{A^2}{B^2}\int \frac{1}{A^2}dt - \int \frac{1}{B^2}dt)\frac{\partial^2 r_9}{\partial x\partial y}]s$$
$$- \frac{\partial r_{13}}{\partial x} \tag{4.45}$$

At this stage μ, τ, ξ, η, φ, and f have the following form:

$$\mu = \frac{\alpha_1}{2}s^2 + \alpha_2 s + \alpha_3, \tag{4.46}$$

$$\tau = \frac{\alpha_2}{2}t + r_{10}(x,y,z) + (\frac{\alpha_1}{2}t + r_9(x,y,z))s, \tag{4.47}$$

$$\xi = (\int \frac{1}{A^2}dt)\frac{\partial r_{10}}{\partial x} + r_{11}(x,y,z) + (\int \frac{1}{A^2}dt)\frac{\partial r_9}{\partial x}s, \tag{4.48}$$

$$\eta = (-\frac{A^2}{B^2}\int \frac{1}{A^2}dt - \int \frac{1}{B^2}dt)x\frac{\partial^2 r_{10}}{\partial x\partial y} + \int \frac{1}{B^2}dt\frac{\partial r_{10}}{\partial y} - \frac{A^2}{B^2}x\frac{\partial r_{11}}{\partial y} + r_{13}(x,y,z) - x\frac{\partial r_{13}}{\partial x}$$
$$+ [(-\frac{A^2}{B^2}\int \frac{1}{A^2}dt - \int \frac{1}{B^2}dt)x\frac{\partial^2 r_9}{\partial x\partial y} + \int \frac{1}{B^2}dt\frac{\partial r_9}{\partial y}]s, \tag{4.49}$$

$$\varphi = (-\frac{A^2}{B^2}\int \frac{1}{A^2}dt - \int \frac{1}{B^2}dt)\frac{\partial^2 r_{10}}{\partial x\partial y} - \frac{A^2}{B^2}\frac{\partial r_{11}}{\partial y} + [(-\frac{A^2}{B^2}\int \frac{1}{A^2}dt - \int \frac{1}{B^2}dt)\frac{\partial^2 r_9}{\partial x\partial y}]s$$
$$- \frac{\partial r_{13}}{\partial x} \tag{4.50}$$

$$f = -\frac{\alpha_1}{2}t^2 - 2r_9(x, y, z)t + r_{12}(x, y, z). \tag{4.51}$$

Differentiating Eq.(4.23) with respect to $'t'$ yields,

$$(1 + AA' \int \frac{1}{A^2}dt)\frac{\partial r_9}{\partial x} = 0. \tag{4.52}$$

From above equation the following 3 possibilities arise:

(a) $(1 + AA' \int \frac{1}{A^2}dt) \neq 0, \quad \frac{\partial r_9}{\partial x} = 0,$

(b) $(1 + AA' \int \frac{1}{A^2}dt) = 0, \quad \frac{\partial r_9}{\partial x} \neq 0,$

(c) $(1 + AA' \int \frac{1}{A^2}dt) = 0, \quad \frac{\partial r_9}{\partial x} = 0.$

Let us first consider the case (a), since $\frac{\partial r_9}{\partial x} = 0$, then $r_9 = r_9(y, z)$, using this result in Eq.(4.23) we get $\frac{\partial r_{12}}{\partial x} = 0$, which means that $r_{12} = r_{12}(y, z)$.

Differentiate Eq.(4.13) with respect to $'s'$ we get

$$2A'(\frac{\alpha_1}{2}t + r_9(y, z)) - \alpha_1 A = 0. \tag{4.53}$$

Differentiating the above equation with respect to $'z'$ implies that $A'\frac{\partial r_9}{\partial z} = 0$, which has the following 3 possibilities:

(i) $A' = 0, \quad \frac{\partial r_9}{\partial z} \neq 0,$

(ii) $A' \neq 0, \quad \frac{\partial r_9}{\partial z} = 0,$

(iii) $A' = 0, \quad \frac{\partial r_9}{\partial z} = 0.$

For case (i): $A' = 0 \Rightarrow A = \beta_1$, where β_1 is constant. Using $A' = 0$ in Eq.(4.53), one immediately finds that $\alpha_1 = 0$.

Now, we can rewrite Eqs.(4.46-4.51) as:

$$\mu = \alpha_2 s + \alpha_3, \tag{4.54}$$

$$\tau = \frac{\alpha_2}{2} t + r_{10}(x, y, z) + r_9(y, z)s, \tag{4.55}$$

$$\xi = \frac{1}{\beta_1^2}\, t\, \frac{\partial r_{10}}{\partial x} + r_{11}(x, y, z), \tag{4.56}$$

$$\eta = (-\frac{1}{B^2}t - \int \frac{1}{B^2}dt)x\frac{\partial^2 r_{10}}{\partial x \partial y} + \int \frac{1}{B^2}dt\frac{\partial r_{10}}{\partial y} - \frac{\beta_1^2}{B^2}x\frac{\partial r_{11}}{\partial y} + \int \frac{1}{B^2}dt\frac{\partial r_9}{\partial y}s - x\frac{\partial r_{13}}{\partial x}$$
$$+ r_{13}(x, y, z), \tag{4.57}$$

$$\varphi = (-\frac{1}{B^2}t - \int \frac{1}{B^2}dt)\frac{\partial^2 r_{10}}{\partial x \partial y} - \frac{\beta_1^2}{B^2}\frac{\partial r_{11}}{\partial y} - \frac{\partial r_{13}}{\partial x}, \tag{4.58}$$

$$f = -2r_9(y, z)t + r_{12}(y, z). \tag{4.59}$$

Continuing to determine the unknowns in the coefficients determining Noether symmetries, now we differentiate Eq.(4.18) with respect to $'s'$ to get

$$x\frac{\partial r_9}{\partial y} + \frac{\partial r_9}{\partial z} = 0, \tag{4.60}$$

the above equation implies that $\frac{\partial r_9}{\partial z} = 0$. This suggests that case (i) does not satisfy Eq.(4.18). Thus this case is excluded.

Now, we will consider the case (a iii): $A = \beta_1$, $\frac{\partial r_9}{\partial z} = 0$, which implies that $r_9 = r_9(y)$.

Put $\frac{\partial r_9}{\partial z}=0$ in Eq.(4.60) we get $\frac{\partial r_9}{\partial y}=0$, and therefore $r_9=\alpha_4$.

Substitute the values of μ and ξ in Eq.(4.13) we obtain

$$\frac{1}{\beta_1^2}t\frac{\partial^2 r_{10}}{\partial x^2}+\frac{\partial r_{11}}{\partial x}=\frac{\alpha_2}{2}. \tag{4.61}$$

Differentiating the above equation with respect to $'t'$ yields $\frac{\partial^2 r_{10}}{\partial x^2}=0$, suggesting that $r_{10}=r_{14}(y,z)x+r_{15}(y,z)$.

Now, substitute $\frac{\partial^2 r_{10}}{\partial x^2}=0$ in Eq.(4.61) we have $\frac{\partial r_{11}}{\partial x}=\frac{\alpha_2}{2}$, and therefore $r_{11}=\frac{\alpha_2}{2}x+r_{16}(y,z)$.

From Eqs.(4.24) and (4.25), we obtain $\frac{\partial r_{12}}{\partial y}=0$ and $\frac{\partial r_{12}}{\partial z}=0$, which means that $r_{12}=\alpha_5$.

Now, the Eqs.(4.54-4.59) take the form:

$$\mu=\alpha_2 s+\alpha_3, \tag{4.62}$$

$$\tau=\frac{\alpha_2}{2}t+r_{14}(y,z)x+r_{15}(y,z)+\alpha_4 s, \tag{4.63}$$

$$\xi=\frac{1}{\beta_1^2}\ r_{14}(y,z)t+\frac{\alpha_2}{2}x+r_{16}(y,z), \tag{4.64}$$

$$\eta=\int\frac{1}{B^2}dt\frac{\partial r_{15}}{\partial y}\ -\frac{1}{B^2}xt\frac{\partial r_{14}}{\partial y}\ -\frac{\beta_1^2}{B^2}x\frac{\partial r_{16}}{\partial y}+r_{13}(x,y,z)-x\frac{\partial r_{13}}{\partial x}, \tag{4.65}$$

$$\varphi=(-\frac{1}{B^2}t-\int\frac{1}{B^2}dt)\frac{\partial r_{14}}{\partial y}-\frac{\beta_1^2}{B^2}\frac{\partial r_{16}}{\partial y}-\frac{\partial r_{13}}{\partial x}, \tag{4.66}$$

$$f=-2\alpha_4 t+\alpha_5. \tag{4.67}$$

Differentiating Eq.(4.15) with respect to $'s'$ leads to

$$\alpha_4(BB'x^2+CC')=0, \tag{4.68}$$

which gives the following 5 possibilities:

(I) $\alpha_4 \neq 0,\ B' = 0\ and\ C' = 0,$

(II) $\alpha_4 = 0,\ B' \neq 0\ and\ C' \neq 0,$

(III) $\alpha_4 = 0,\ \ B' = 0\ and\ C' \neq 0,$

(IV) $\alpha_4 = 0,\ \ B' \neq 0\ and\ C' = 0,$

(V) $\alpha_4 = 0,\ \ B' = 0\ and\ C' = 0.$

Now, consider the case (a iii I):

$A = \beta_1,\ B = \beta_2,\ and\ C = \beta_3$, where $\beta_1,\ \beta_2\ and\ \beta_3$ are constants.

From Eq.(4.15) we get

$$\begin{aligned}&2xt\tfrac{\partial^2 r_{15}}{\partial y\partial z} + 2xt^2\tfrac{\partial^2 r_{14}}{\partial y\partial z} + 2\beta_2^2 x\tfrac{\partial r_{13}}{\partial z} + \beta_3^2\alpha_2 + 2\tfrac{\beta_1^2\beta_3^2}{\beta_2^2}\tfrac{\partial^2 r_{16}}{\partial y\partial z}\\ &+2\beta_3^2\tfrac{\partial^2 r_{13}}{\partial x\partial z} + 4\tfrac{\beta_3^2}{\beta_2^2}t\tfrac{\partial^2 r_{14}}{\partial y\partial z} - 2\tfrac{\beta_2^2}{\beta_1^2}xt\, r_{14}(y,z) - 2\beta_2^2 x\, r_{14}(y,z) = 0,\end{aligned} \tag{4.69}$$

differentiate the above equation with respect to $'x'$ and then with respect to $'t'$ two times, we obtain $\frac{\partial^2 r_{14}}{\partial y\partial z} = 0,$ which implies that $r_{14} = r_{17}(y) + r_{18}(z)$.

Substitute the value of r_{14} in Eq.(4.69) and then differentiate with respect to $'x', 't'$ we get

$$\frac{\partial^2 r_{15}}{\partial y\partial z} - \frac{\beta_2^2}{\beta_1^2}r_{17}(y) - \frac{\beta_2^2}{\beta_1^2}r_{18}(z) = 0, \tag{4.70}$$

now, differentiate the above equation with respect to $'y', 'z'$, we obtain $\frac{\partial^4 r_{15}}{\partial y^2\partial z^2} = 0$, and therefore $r_{15} = yr_{21}(z) + r_{22}(z) + zr_{19}(y) + r_{20}(y)$.

Put the value of r_{15} in Eq.(4.70) we get

$$\frac{\partial r_{21}}{\partial z} + \frac{\partial r_{19}}{\partial y} = \frac{\beta_2^2}{\beta_1^2}r_{17}(y) + \frac{\beta_2^2}{\beta_1^2}r_{18}(z), \tag{4.71}$$

differentiate the above equation with respect to $'y'$ we obtain $\frac{\partial^2 r_{19}}{\partial y^2} = \frac{\beta_2^2}{\beta_1^2}\frac{\partial r_{17}}{\partial y}$, which implies that $r_{19}(y) = \frac{\beta_2^2}{\beta_1^2}\int r_{17}(y)dy + \alpha_6 y + \alpha_7$. Differentiating Eq.(4.71) with respect to $'z'$ gives $r_{21}(z) = \frac{\beta_2^2}{\beta_1^2}\int r_{18}(z)dz + \alpha_8 z + \alpha_9$.

Substituting the values of r_{19} and r_{21} in Eq.(4.71) gives $\alpha_6 = -\alpha_8$.

Now Eqs.(4.62-4.67) take the form:

$$\mu = \alpha_2 s + \alpha_3, \tag{4.72}$$

$$\tau = x(r_{17}(y) + r_{18}(z)) + \frac{\beta_2^2}{\beta_1^2}y\int r_{18}(z)dz + \frac{\beta_2^2}{\beta_1^2}z\int r_{17}(y)dy + r_{22}(z) + r_{20}(y) + \frac{\alpha_2}{2}t + \alpha_7 z + \alpha_9 y + \alpha_4 s, \tag{4.73}$$

$$\xi = \frac{1}{\beta_1^2}t(r_{17}(y) + r_{18}(z)) + r_{16}(y,z) + \frac{\alpha_2}{2}x, \tag{4.74}$$

$$\eta = \frac{1}{\beta_2^2}t(\frac{\beta_2^2}{\beta_1^2}\int r_{18}(z)dz + \frac{\beta_2^2}{\beta_1^2}zr_{17}(y) + \frac{\partial r_{20}}{\partial y} + \alpha_9) - \frac{1}{\beta_2^2}xt\frac{\partial r_{17}}{\partial y} - \frac{\beta_1^2}{\beta_2^2}x\frac{\partial r_{16}}{\partial y} - x\frac{\partial r_{13}}{\partial x} r_{13}(x,y,z), \tag{4.75}$$

$$\varphi = -\frac{\beta_1^2}{\beta_2^2}\frac{\partial r_{16}}{\partial y} - \frac{\partial r_{13}}{\partial x} - \frac{2}{\beta_2^2}t\frac{\partial r_{17}}{\partial y}, \tag{4.76}$$

$$f = -2\alpha_4 t + \alpha_5. \tag{4.77}$$

At this stage we substitute the value of μ, τ, η, ϕ into Eq.(4.14) to get:

$$2\frac{\beta_2^2}{\beta_1^2}tz\frac{\partial r_{17}}{\partial y}+2xt\frac{\partial^2 r_{17}}{\partial y^2}+2\beta_2^2\frac{\partial r_{13}}{\partial y}+2t\frac{\partial^2 r_{20}}{\partial y^2}-\beta_2^2\alpha_2=0, \tag{4.78}$$

differentiating above equation with respect to $'t'$, $'z'$ we obtain $\frac{\partial r_{17}}{\partial y}=0$, this suggests that $r_{17}=\alpha_{10}$. Thus, Eq.(4.78) becomes:

$$2\beta_2^2\frac{\partial r_{13}}{\partial y}+2t\frac{\partial^2 r_{20}}{\partial y^2}-\beta_2^2\alpha_2=0. \tag{4.79}$$

Differentiating Eq.(4.79) with respect to $'t'$ yields $\frac{\partial^2 r_{20}}{\partial y^2}=0$, which implies that $r_{20}(y)=\alpha_{11}y+\alpha_{12}$.

Substituting the value of $r_{20}(y)$ into Eq.(4.79) gives $\frac{\partial r_{13}}{\partial y}=\frac{\alpha_2}{2}$, and therefore $r_{13}(x,y,z)=\frac{\alpha_2}{2}y+r_{23}(x,z)$.

Now, we can rewrite Eqs.(4.72-4.77) as:

$$\mu=\alpha_2 s+\alpha_3, \tag{4.80}$$

$$\tau=xr_{18}(z)+\frac{\beta_2^2}{\beta_1^2}y\int r_{18}(z)dz+r_{22}(z)+\frac{\beta_2^2}{\beta_1^2}\alpha_{10}yz+\alpha_{10}x+\frac{\alpha_2}{2}t+\alpha_7 z+\alpha_4 s+\alpha_{12} \\ +(\alpha_9+\alpha_{11})y, \tag{4.81}$$

$$\xi = \frac{1}{\beta_1^2} t(\alpha_{10} + r_{18}(z)) + r_{16}(y,z) + \frac{\alpha_2}{2} x, \tag{4.82}$$

$$\eta = \frac{1}{\beta_2^2} t(\frac{\beta_2^2}{\beta_1^2} \int r_{18}(z)dz + \frac{\beta_2^2}{\beta_1^2}\alpha_{10} z + \alpha_9 + \alpha_{11}) - \frac{\beta_1^2}{\beta_2^2} x \frac{\partial r_{16}}{\partial y} + r_{23}(x,z) - x\frac{\partial r_{23}}{\partial x} + \frac{\alpha_2}{2} y, \tag{4.83}$$

$$\varphi = -\frac{\beta_1^2}{\beta_2^2}\frac{\partial r_{16}}{\partial y} - \frac{\partial r_{23}}{\partial x}, \tag{4.84}$$

$$f = -2\alpha_4 t + \alpha_5. \tag{4.85}$$

From Eq.(4.18) we obtain

$$\begin{aligned} &\tfrac{\beta_2^2}{\beta_1^2} x \int r_{18}(z)dz + \tfrac{\beta_2^2}{\beta_1^2} y r_{18}(z) + x\tfrac{\partial r_{18}}{\partial z} + \tfrac{\partial r_{22}}{\partial z} + \tfrac{\beta_2^2}{\beta_1^2}\alpha_{10} xz \\ &+ \tfrac{\beta_2^2}{\beta_1^2}\alpha_{10} y + (\alpha_9 + \alpha_{11})x + \alpha_7 = 0, \end{aligned} \tag{4.86}$$

we noe differentiate above equation with respect to $'y'$ and obtain $r_{18} = -\alpha_{10}$. In the light of above calculations, Eq.(4.86) becomes:

$$\frac{\partial r_{22}}{\partial z} + (\alpha_9 + \alpha_{11})x + \alpha_7 = 0, \tag{4.87}$$

now, differentiating above equation with respect to $'x'$ yields $\alpha_9 = -\alpha_{11}$. using this in Eq.(4.87) we get $\frac{\partial r_{22}}{\partial z} = -\alpha_7$, which implies that $r_{22}(z) = -\alpha_7 z + \alpha_{13}$. Now, Eqs.(4.80-4.85) have the following form:

$$\mu = \alpha_2 s + \alpha_3, \tag{4.88}$$

$$\tau = \frac{\alpha_2}{2} t + \alpha_4 s + \alpha_{14}, \tag{4.89}$$

$$\xi = \frac{\alpha_2}{2} x + r_{16}(y,z), \tag{4.90}$$

$$\eta = -\frac{\beta_1^2}{\beta_2^2} x \frac{\partial r_{16}}{\partial y} + r_{23}(x,z) - x\frac{\partial r_{23}}{\partial x} + \frac{\alpha_2}{2} y, \tag{4.91}$$

$$\varphi = -\frac{\beta_1^2}{\beta_2^2} \frac{\partial r_{16}}{\partial y} - \frac{\partial r_{23}}{\partial x}, \tag{4.92}$$

$$f = -2\alpha_4 t + \alpha_5. \tag{4.93}$$

In order to determine the remaining unknowns in Eqs.(4.90-4.92), we proceed as follows:

substituting the values of ξ, η, ϕ into Eq.(4.20) leads to

$$\beta_1^2 x \frac{\partial r_{16}}{\partial y} + \beta_1^2 \frac{\partial r_{16}}{\partial z} - \beta_3^2 \frac{\partial^2 r_{23}}{\partial x^2} = 0, \tag{4.94}$$

Differentiate above equation with respect to $'x', 'y'$ we get $\frac{\partial^2 r_{16}}{\partial y^2} = 0$, which implies that $r_{16}(y,z) = r_{24}(z) y + r_{25}(z)$.

Also, differentiating Eq.(4.94) with respect to $'y'$ gives $\frac{\partial^2 r_{16}}{\partial y \partial z} = 0$, implying that $\frac{\partial r_{24}}{\partial z} = 0$, giving $r_{24} = \alpha_{15}$.

Using the value of r_{16} in Eq.(4.94) and then differentiating the resulting equation with respect to $'x'$ we obtain, $\frac{\partial^3 r_{23}}{\partial x^3} = \frac{\beta_1^2}{\beta_3^2}\alpha_{15}$, which implies that $r_{23}(x,z) = \frac{\beta_1^2}{\beta_3^2}\alpha_{15}\frac{x^3}{6} + r_{26}(z)\frac{x^2}{2} + r_{27}(z)x + r_{28}(z)$.

In the light of above calculations we can rewrite Eqs.(4.88-4.93) as follows:

$$\mu = \alpha_2 s + \alpha_3, \tag{4.95}$$

$$\tau = \frac{\alpha_2}{2} t + \alpha_4 s + \alpha_{14}, \tag{4.96}$$

$$\xi = r_{25}(z) + \frac{\alpha_2}{2}x + \alpha_{15}y, \tag{4.97}$$

$$\eta = -\frac{x^2}{2}\, r_{26}(z) + r_{28}(z) - \frac{\beta_1^2}{\beta_3^2}\,\alpha_{15}\,\frac{x^3}{3} - \frac{\beta_1^2}{\beta_3^2}\,\alpha_{15}\,x + \frac{\alpha_2}{2}y, \tag{4.98}$$

$$\varphi = -x\, r_{26}(z) - r_{27}(z) - \frac{\beta_1^2}{\beta_3^2}\,\alpha_{15}\,\frac{x^2}{2} - \frac{\beta_1^2}{\beta_3^2}\,\alpha_{15}, \tag{4.99}$$

$$f = -2\alpha_4 t + \alpha_5. \tag{4.100}$$

At this stage we substitute values of $\mu, \tau, \xi, \eta, \text{and}\, \phi$ into Eq.(4.21) to get:

$$x^2\frac{\partial r_{26}}{\partial z} - 2r_{25}(z) + 2\frac{\partial r_{28}}{\partial z} + 2x\frac{\partial r_{27}}{\partial z} - 2\alpha_{15}y = 0, \tag{4.101}$$

differentiating above equation with respect to $'y'$ once and $'x'$ twice respectively gives, $\alpha_{15} = 0$ and $r_{26} = \alpha_{16}$. Substituting the values of r_{26} and α_{15} into Eq.(4.101) and then differentiating the resulting equation with respect to $'x'$ yields $r_{27} = \alpha_{17}$. Now, Eq.(4.101) reduces to $\frac{\partial r_{28}}{\partial z} = r_{25}(z)$, which implies that $r_{28}(z) = \int r_{25}(z)dz + \alpha_{18}$.

At this stage, the Eqs.(4.95-4.100) take the form:

$$\mu = \alpha_2 s + \alpha_3, \tag{4.102}$$

$$\tau = \frac{\alpha_2}{2}t + \alpha_4 s + \alpha_{14}, \tag{4.103}$$

$$\xi = r_{25}(z) + \frac{\alpha_2}{2}x, \tag{4.104}$$

$$\eta = \int r_{25}(z)dz - \alpha_{16}\frac{x^2}{2} + \frac{\alpha_2}{2}y + \alpha_{18}, \tag{4.105}$$

$$\varphi = -\alpha_{16}\, x - \alpha_{17}, \tag{4.106}$$

$$f = -2\alpha_4 t + \alpha_5. \tag{4.107}$$

It is clear from the above set that we are left with only one unknown $r_{25}(z)$, to determine this, we proceed as follows:

from Eq.(4.15), we obtain $\beta_3^2\,\alpha_2 = 0$, implying that $\alpha_2 = 0$. And from Eq.(4.20) we have $\beta_1^2 \frac{\partial r_{25}}{\partial z} - \beta_3^2 \alpha_{16} = 0$, which implies that $r_{25}(z) = \frac{\beta_3^2}{\beta_1^2}\alpha_{16} z + \alpha_{19}$.

Finally, we obtain the following form for $\mu, \tau, \xi, \eta,\, \phi$ and f:

$$\mu = \alpha_3, \tag{4.108}$$

$$\tau = \alpha_4 s + \alpha_{14}, \tag{4.109}$$

$$\xi = \frac{\beta_3^2}{\beta_1^2}\alpha_{16} z + \alpha_{19}, \tag{4.110}$$

$$\eta = \frac{\beta_3^2}{\beta_1^2}\alpha_{16}\frac{z^2}{2} - \alpha_{16}\frac{x^2}{2} + \alpha_{19} z + \alpha_{18}, \tag{4.111}$$

$$\varphi = -\alpha_{16}\, x - \alpha_{17}, \tag{4.112}$$

$$f = -2\alpha_4 t + \alpha_5. \tag{4.113}$$

Therefore, we obtain the following Noether symmetries and adjoint representations for the case(a ii III):

Case 1:

$A' = 0, B' = 0,\, and\, C' = 0.$

The *Noether symmetries* are:

$X_1 = \frac{\partial}{\partial s}, X_2 = s\frac{\partial}{\partial t}; f = -2t, X_3 = z\frac{\partial}{\partial x} - \frac{x^2}{2}\frac{\partial}{\partial y} + \frac{z^2}{2}\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$

$X_4 = \frac{\partial}{\partial t}, X_5 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, X_6 = \frac{\partial}{\partial y}, X_7 = -\frac{\partial}{\partial z}.$

The *commutator table* for these symmetries is given by

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	X_4	0	0	0	0	0
X_2	$-X_4$	0	0	0	0	0	0
X_3	0	0	0	0	$-X_7$	0	X_5
X_4	0	0	0	0	0	0	0
X_5	0	0	X_7	0	0	0	X_6
X_6	0	0	0	0	0	0	0
X_7	0	0	$-X_5$	0	$-X_6$	0	0

Table 4.1: The commutator table for the case 1

From the above table, one can find the *adjoint representations* for these symmetries, these representations are given by the following matrices:

$$ad(X_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad ad(X_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad ad(X_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad ad(X_6) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_7) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the classification of the Noether symmetries, several cases arise. While solving for these symmetries, some of the cases are either leading to inconsistent system or being a special cases of the others. All such cases are excluded. There is a total of 13 cases for which the Noether symmetries are calculated. In the rest of this chapter we only list our main results without giving the detailed calculations.

Case 2:

$A' = 0, B' = 0, C' \neq 0 \; and \; C'' = 0,$

the values of $\mu, \tau, \xi, \eta, \phi$ and f are:

$$\mu = \alpha_2 s + \alpha_3, \tag{4.114}$$

$$\tau = \frac{\alpha_2}{2} t + \alpha_6, \tag{4.115}$$

$$\xi = \frac{\alpha_2}{2} x + \alpha_7, \tag{4.116}$$

$$\eta = \frac{\alpha_2}{2} y + \alpha_7 z + \alpha_8, \tag{4.117}$$

$$\varphi = -\alpha_9, \tag{4.118}$$

$$f = \alpha_5. \tag{4.119}$$

The obtained *Noether symmetries* are:

$X_1 = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}, X_2 = \frac{\partial}{\partial s},$

$X_3 = \frac{\partial}{\partial t}, X_4 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, X_5 = \frac{\partial}{\partial y}, X_6 = -\frac{\partial}{\partial z}.$

The *commutator table* for these symmetries is:

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$-X_2$	$-\frac{1}{2}X_3$	$-\frac{1}{2}X_4$	$-\frac{1}{2}X_5$	0
X_2	X_2	0	0	0	0	0
X_3	$\frac{1}{2}X_3$	0	0	0	0	0
X_4	$\frac{1}{2}X_4$	0	0	0	0	X_5
X_5	$\frac{1}{2}X_5$	0	0	0	0	0
X_6	0	0	0	$-X_5$	0	0

Table 4.2: The commutator table for the case 2

The *adioint representations* for these symmetries are:

$$ad(X_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad ad(X_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad ad(X_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad ad(X_6) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Case 3:

$A' = 0, B' = 0, C' \neq 0$ *and* $C'' \neq 0$,

the values of $\mu, \tau, \xi, \eta, \phi$ and f are given by

$$\mu = \alpha_3, \tag{4.120}$$

$$\tau = 0, \tag{4.121}$$

$$\xi = \alpha_7, \tag{4.122}$$

$$\eta = \alpha_7 z + \alpha_8, \tag{4.123}$$

$$\varphi = -\alpha_9, \tag{4.124}$$

$$f = \alpha_5. \tag{4.125}$$

The *Noether symmetries* are:

$X_1 = \frac{\partial}{\partial s}, X_2 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, X_3 = \frac{\partial}{\partial y}, X_4 = -\frac{\partial}{\partial z}.$

The *commutator table* for these symmetries is:

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	X_3
X_3	0	0	0	0
X_4	0	$-X_3$	0	0

Table 4.3: The commutator table for the case 3

The *adjoint representations* for these symmetries are:

$$ad(X_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad ad(X_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad ad(X_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case 4:

$A' = 0, B' \neq 0$ *and* $C' = 0,$

the values of $\mu, \tau, \xi, \eta, \phi$ and f are given by

$$\mu = \alpha_3, \tag{4.126}$$

$$\tau = 0, \tag{4.127}$$

$$\xi = \alpha_7 z + \alpha_8, \tag{4.128}$$

$$\eta = -\alpha_7 \frac{x^2}{2} + \alpha_7 \frac{z^2}{2} + \alpha_8 z + \alpha_9, \tag{4.129}$$

$$\phi = -\alpha_7 x - \alpha_{10}, \tag{4.130}$$

$$f = \alpha_5. \tag{4.131}$$

The *Noether symmetries* are:

$X_1 = \frac{\partial}{\partial s}, X_2 = z\frac{\partial}{\partial x} - \frac{x^2}{2}\frac{\partial}{\partial y} + \frac{z^2}{2}\frac{\partial}{\partial y} - x\frac{\partial}{\partial z},$

$X_3 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, X_4 = \frac{\partial}{\partial y}, X_5 = -\frac{\partial}{\partial z}.$

The commutator table for these symmetries is:

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	0
X_2	0	0	$-X_5$	0	X_3
X_3	0	X_5	0	0	X_4
X_4	0	0	0	0	0
X_5	0	$-X_3$	$-X_4$	0	0

Table 4.4: The commutator table for the case 4

The *adjoint representations* for these symmetries are:

$$ad(X_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad ad(X_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$ad(X_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad ad(X_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case 5:

$A' = 0, B' \neq 0$ *and* $C' \neq 0$,

It is similar to the *case 3*.

Case 6:

$A' \neq 0, A'' = 0,$ *and* $A = B = C,$

the values of $\mu, \tau, \xi, \eta, \phi$ and f are given by

$$\mu = \alpha_1 \frac{s^2}{2} + \alpha_2 s + \alpha_3, \tag{4.132}$$

$$\tau = \frac{\alpha_1}{2} st + \frac{\alpha_2}{2} t + \alpha_4 s + \alpha_6, \tag{4.133}$$

$$\xi = \alpha_7, \tag{4.134}$$

$$\eta = \alpha_7 z + \alpha_8, \tag{4.135}$$

$$\phi = -\alpha_9, \tag{4.136}$$

$$f = -\frac{\alpha_1}{2} t^2 - 2\alpha_4 t + \alpha_5. \tag{4.137}$$

The *Noether symmetries* are:

$X_1 = \frac{s^2}{2}\frac{\partial}{\partial s} + \frac{ts}{2}\frac{\partial}{\partial t}; f = -\frac{t^2}{2},$

$X_2 = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t}, X_3 = \frac{\partial}{\partial t}, X_4 = \frac{\partial}{\partial s}, X_5 = s\frac{\partial}{\partial t}; f = -2t$

$X_6 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, X_7 = \frac{\partial}{\partial y}, X_8 = -\frac{\partial}{\partial z}.$

The *commutator table* for these symmetries is given by

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	$-X_1$	$\frac{1}{2}X_5$	$-X_2$	0	0	0	0
X_2	X_1	0	$-\frac{1}{2}X_3$	$-X_4$	X_5	0	0	0
X_3	$-\frac{1}{2}X_5$	$\frac{1}{2}X_3$	0	0	0	0	0	0
X_4	X_2	X_4	0	0	X_3	0	0	0
X_5	0	$-X_5$	0	$-X_3$	0	0	0	0
X_6	0	0	0	0	0	0	0	X_7
X_7	0	0	0	0	0	0	0	0
X_8	0	0	0	0	0	$-X_7$	0	0

Table 4.5: The commutator table for the case 6

The *adjoint representations* for these symmetries are:

$$ad(X_1) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_6) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_7) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_8) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Case 7:

$A' \neq 0, A'' \neq 0, B'' = 0$ *and* $C'' = 0,$

the values of $\mu, \tau, \xi, \eta, \phi$ and f are given by

$$\mu = \alpha_2 s + \alpha_3, \tag{4.138}$$

$$\tau = \frac{\alpha_2}{2} t + \alpha_6, \tag{4.139}$$

$$\xi = 0, \tag{4.140}$$

$$\eta = \alpha_7, \tag{4.141}$$

$$\phi = -\alpha_8, \tag{4.142}$$

$$f = \alpha_5. \tag{4.143}$$

The *Noether symmetries* are:

$X_1 = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial s},$

$X_3 = \frac{\partial}{\partial t}, X_4 = \frac{\partial}{\partial y}, X_5 = -\frac{\partial}{\partial z}.$

The *commutator table* for these symmetries is given by

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-X_2$	$-\frac{1}{2}X_3$	0	0
X_2	X_2	0	0	0	0
X_3	$\frac{1}{2}X_3$	0	0	0	0
X_4	0	0	0	0	0
X_5	0	0	0	0	0

Table 4.6: The commutator table for the case 7

The *adjoint representations* for these symmetries are:

$$ad(X_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad ad(X_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad ad(X_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Case 8:

$A' \neq 0, A'' \neq 0, B' = 0$ *and* $C'' = 0,$

the values of $\mu, \tau, \xi, \eta, \phi$ and f are given by

$$\mu = \alpha_3, \tag{4.144}$$

$$\tau = \alpha_6, \tag{4.145}$$

$$\xi = 0, \tag{4.146}$$

$$\eta = \alpha_7, \tag{4.147}$$

$$\phi = -\alpha_8, \tag{4.148}$$

$$f = \alpha_5. \tag{4.149}$$

The *Noether symmetries* are:

$X_1 = \frac{\partial}{\partial s}, X_2 = \frac{\partial}{\partial t}, X_3 = \frac{\partial}{\partial y}, X_4 = -\frac{\partial}{\partial z}.$

The *commutator table* for these symmetries is given by

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	0
X_3	0	0	0	0
X_4	0	0	0	0

Table 4.7: The commutator table for the case 8

The *adjoint representations* for these symmetries are:

$$ad(X_1) = ad(X_2) = ad(X_3) = ad(X_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Case 9:

$A' \neq 0, A'' \neq 0, B = B(t)$ *and* $C'' = 0,$

the values of $\mu, \tau, \xi, \eta, \phi$ and f are given by

$$\mu = \alpha_3, \tag{4.150}$$

$$\tau = 0, \tag{4.151}$$

$$\xi = 0, \tag{4.152}$$

$$\eta = \alpha_7, \tag{4.153}$$

$$\phi = -\alpha_8, \tag{4.154}$$

$$f = \alpha_5. \tag{4.155}$$

The *Noether symmetries* are:

$X_1 = \frac{\partial}{\partial s}, X_2 = \frac{\partial}{\partial y}, X_3 = -\frac{\partial}{\partial z}.$

The *commutator table* for these symmetries is given by

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	0
X_3	0	0	0

Table 4.8: The commutator table for the case 9

The *adjoint representations* for these symmetries are:

$$ad(X_1) = ad(X_2) = ad(X_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Case 10:

$A' \neq 0, A'' \neq 0, B = B(t) \ and \ C'' \neq 0,$

it is similar to the case 9.

Case 11:

$A' \neq 0, A'' = 0, B = (a_1^2 t^2 + 2a_1 a_2 t - a_2^2)^{\frac{-1}{2}} \ and \ C = C(t),$

it is similar to the case 9.

Case 12:

$A' \neq 0, A'' = 0, B'' \neq 0 \ and \ C = C(t),$

it is similar to the case 9.

Case 13:

$A' \neq 0, A'' = 0, B' = 0, C = C(t),$

the values of $\mu, \tau, \xi, \eta, \phi$ and f are given by

$$\mu = \alpha_2 s + \alpha_3, \tag{4.156}$$

$$\tau = \alpha_2(\frac{t}{2} + 1), \tag{4.157}$$

$$\xi = 0, \tag{4.158}$$

$$\eta = \alpha_6 y + \alpha_7, \tag{4.159}$$

$$\phi = \alpha_6 z - \alpha_8, \tag{4.160}$$

$$f = \alpha_5. \tag{4.161}$$

The *Noether symmetries* are:

$X_1 = s\frac{\partial}{\partial s} + (\frac{t}{2} + 1)\frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial s}$

$X_3 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, X_4 = \frac{\partial}{\partial y}, X_5 = -\frac{\partial}{\partial z}.$

The *commutator table* for these symmetries is given by

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-X_2$	0	0	0
X_2	X_2	0	0	0	0
X_3	0	0	0	$-X_4$	$-X_5$
X_4	0	0	X_4	0	0
X_5	0	0	X_5	0	0

Table 4.9: The commutator table for the case 13

The *adjoint representations* for these symmetries are:

$$ad(X_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad ad(X_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad ad(X_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(X_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

4.2 The Killing Form and Classification of the Lie Algebras

In this section we find the killing form for the Noether symmetries obtained in the previous section, and classify the type of the Lie algebras of these symmetries.

Case 1:

Let $L_1 = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$ be the Lie algebra of symmetries obtained from *case 1*, then:

$\kappa(X_3, X_3) = -2, \kappa(X_i, X_j) = 0, \mathrm{i} = 1,...,7$ and j = 1,2,4,5,6,7.

The matrix of κ then:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the ij-entry of the matrix is $\kappa(X_i, X_j)$.

The center of L_1 is given by:

$$\begin{aligned} Z(L_1) &= \{x \in L_1 : [x, y] = 0 \ for \, all \, y \in L_1\} \\ &= \{X_4, X_6\} \end{aligned}.$$

The radical of L_1 is given by:

$$\Re(L_1) = \{x \in L_1 : k(x,y) = 0 \;\; for\, all\, y \in L_1\}$$
$$= \{X_1, X_2, X_4, X_5, X_6, X_7\}$$

The derived series of L_1 is the series with term:

$$L_1^{(0)} = L_1 \quad and \quad L_1^{(k)} = [L_1^{(k-1)}, L_1^{(k-1)}], k \geq 1.$$

$$Then\ L_1 \supseteq L_1^{(1)} \supseteq L_1^{(2)} \supseteq ...$$

$$L_1^{(1)} = [L_1, L_1] = \{aX_4, bX_5, cX_6, dX_7\}.\, a,b,c,d \in R$$

$$L_1^{(2)} = [L_1^{(1)}, L_1^{(1)}] = \{mX_6\}.\ m \in R$$

$$L_1^{(3)} = [L_1^{(2)}, L_1^{(2)}] = \{0\}.$$

Therefore, we conclude that L_1 is *solvable*.

Case 2:

Let $L_2 = \{X_1, X_2, X_3, X_4, X_5, X_6\}$, then:

$\kappa(X_1, X_1) = 1.75, \kappa(X_i, X_j) = 0,$ i = 1,...,6 and j = 2,3,4,5,6.

The matrix of κ then:

$$\begin{pmatrix} 1.75 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The center of L_2 is given by

$Z(L_2) = \{0\}.$

The radical of L_2 is given by

$\Re(L_2) = \{X_2, X_3, X_4, X_5, X_6\}$

The derived series of L_2 is given by

$L_2^{(1)} = [L_2, L_2] = \{aX_2, bX_3, cX_4, dX_5\}. \, a, b, c, d \in R$

$L_2^{(2)} = [L_2^{(1)}, L_2^{(1)}]$

$= \{0\}.$

Therefore, L_2 is *solvable.*

Case 3:

Let $L_3 = \{X_1, X_2, X_3, X_4\}$, then:

$\kappa(X_i, X_j) = 0$, i = 1,...,4 and j = 1,...,4.

The matrix of κ then:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The center of L_3 is given by

$Z(L_3) = \{X_1, X_3\}$

The radical of L_3 is given by

$\Re(L_3) = L_3.$

The derived series of L_3 is given by

$L_3^{(1)} = [L_3, L_3] = \{aX_3\}. \, a \in R$

$L_3^{(2)} = [L_3^{(1)}, L_3^{(1)}] = \{0\}$

Therefore, L_3 is *solvable.*

Case 4:

Let $L_4 = \{X_1, X_2, X_3, X_4, X_5\}$, then:

$\kappa(X_2, X_2) = -2$, $\kappa(X_i, X_j) = 0$, i = 1,...,5 and j = 1,3,4,5.

The matrix of κ then:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The center of L_4 is given by

$Z(L_4) = \{X_1, X_4\}$

The radical of L_4 is given by

$\Re(L_4) = \{X_1, X_3, X_4, X_5\}$.

The derived series of L_4 is given by

$L_4^{(1)} = [L_4, L_4] = \{aX_3, bX_4, cX_5\}.\ a, b, c \in R$

$L_4^{(2)} = [L_4^{(1)}, L_4^{(1)}] = \{dX_4\}.\ d \in R$.

$L_4^{(3)} = [L_4^{(2)}, L_4^{(2)}] = \{0\}$

Therefore, L_4 is *solvable*.

Case 5:

it is similar to the *case 3*.

Case 6:

Let $L_6 = \{X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8\}$, then:

$\kappa(X_1, X_4) = -1.5, \kappa(X_1, X_j) = 0,\ j = 2, 3, ..., 8,$

$\kappa(X_2, X_2) = 3.25, \kappa(X_2, X_i) = 0,\ i = 3, ..., 8,$

$\kappa(X_s, X_m) = 0,\ s = 3, ..., 8,\ m = 3, ..., 8$

The matrix of κ then:

$$\begin{pmatrix} 0 & 0 & 0 & -1.5 & 0 & 0 & 0 & 0 \\ 0 & 3.25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The center of L_6 is given by

$Z(L_6) = \{X_7\}$

The radical of L_6 is given by

$\Re(L_6) = \{X_3, X_5, X_6, X_7, X_8\}.$

The derived series of L_6 is given by

$L_6^{(1)} = [L_6, L_6] = \{aX_1, bX_2, cX_3, dX_4, eX_5, fX_7\}. a, b, c, d, e, f \in R$

$L_6^{(2)} = [L_6^{(1)}, L_6^{(1)}] = \{jX_1, hX_2, mX_3, nX_4, rX_5\}.\ j, h, m, n, r \in R$

$L_6^{(3)} = [L_6^{(2)}, L_6^{(2)}] = L_6^{(2)}$

$L_6^{(k)} = L_6^{(2)}, \quad k \geq 4.$

Therefore, L_6 is not *solvable*.

Case 7:

Let $L_7 = \{X_1, X_2, X_3, X_4, X_5\}$, then:

$\kappa(X_1, X_1) = 1.25, \kappa(X_i, X_j) = 0,\ i = 1, ..., 5\ j = 2, 3, 4, 5.$

The matrix of κ then:

$$\begin{pmatrix} 1.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The center of L_7 is given by

$Z(L_7) = \{X_4, X_5\}$

The radical of L_7 is given by

$\Re(L_7) = \{X_2, X_3, X_4, X_5\}.$

The derived series of L_6 is given by

$L_7^{(1)} = [L_7, L_7] = \{bX_2, cX_3\}. b, c \in R$

$L_7^{(2)} = [L_7^{(1)}, L_7^{(1)}] = \{0\}$

Therefore, L_7 is *solvable*.

Case 8:

Let $L_8 = \{X_1, X_2, X_3, X_4\}$, then:

$\kappa(X_i, X_j) = 0$, $i = 1, ..., 4\, j = 1, ..., 4$.

The matrix of κ then:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The center of L_8 is given by

$Z(L_8) = L_8$

The radical of L_8 is given by

$\Re(L_8) = L_8$.

The derived series of L_8 is given by

$L_8^{(k)} = \{0\}, \;\; k \geq 1$.

Therefore, L_8 is *solvable.*

Case 9:

Let $L_9 = \{X_1, X_2, X_3\}$, then:

$\kappa(X_i, X_j) = 0$, $i = 1, 2, 3\, j = 1, 2, 3$.

The matrix of κ then:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The center of L_9 is given by

$Z(L_9) = L_9$

The radical of L_8 is given by

$\Re(L_9) = L_9$.

The derived series of L_8 is given by

$L_9^{(k)} = \{0\}, \;\; k \geq 1.$

Therefore, L_9 is *solvable.*

Case 10:

it is similar to the *case 9.*

Case 11:

it is similar to the *case 9.*

Case 12:

it is similar to the *case 9.*

Case 13:

Let $L_13 = \{X_1, X_2, X_3, X_4, X_5\}$, then:

$\kappa(X_1, X_1) = 1, \kappa(X_3, X_3) = 2, \kappa(X_i, X_j) = 0, \; i = 1, ..., 5; \; j = 2, 4, 5.$

The matrix of κ then:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The center of L_{13} is given by

$Z(L_{13}) = \{0\}$

The radical of L_{13} is given by

$\Re(L_{13}) = \{X_2, X_4, X_5\}.$

The derived series of L_{13} is given by

$$L_{13}^{(1)} = [L_{13}, L_{13}] = \{bX_2, cX_4, dX_5\}.\, b, c, d \in R$$
.

$$L_{13}^{(2)} = [L_{13}^{(1)}, L_{13}^{(1)}] = \{0\}$$

Therefore, L_{13} is *solvable*.

CHAPTER 5

CONCLUSION AND RECOMMENDATIONS

In this thesis we have obtained a complete classification of the Noether symmetries of a Lagrangian that is constructed from the the Bianchi type II metrics. This classification of Noether symmetries comes from a total of twenty five differential constraints which are satisfied by the coefficients $A(t)$, $B(t)$, and $C(t)$ appearing in the Bianchi II spacetime metric. To find the Noether symmetries we have solve all the determining equations by requiring consistency of the differential constraints and the equations. In this process it is found that out of twenty five only thirteen cases survive for which the Noetehr symmetries are obtainable.

For each of the thirteen cases a corresponding Noether symmetry group is obtained, of which the smallest one is a 3-parameter Noether symmetry group G_3, while the largest is of eight parameters G_8. Also, it is easily seen that $G_3 \subset G_4' \subset G_5' \subset G_7$, $G_3 \subset G_4'' \subset G_5'' \subset G_8$, $G_3 \subset G_4' \subset G_6$ and $G_3 \subset G_4'' \subset G_6$, whereas G'_4 is a 4-parameter group obtained from *case 3* , G''_4 a 4-parameter group obtained from *case 8* , G'_5 a 5-parameter group obtained from *case 5* , while G''_5 represents a 5-parameter Noether symmetry group given by *case 7*. Further, it is found that for the G_8, the metric coefficients satisfy an explicit linear first order constraint given by $A = B = C = a_1 t + a_2$.

In order to understand the solvability or otherwise of the Noether symmetry groups, we have worked out the adjoint representations and the killing forms in each case. Using these, we have identified the type of the Lie algebras of the Noether symmetries in each case. Interestingly, it is found that all of the Lie algebras of the Noether symmetries are solvable except the one in which the Noether symmetry group is maximal i.e., G_8.

For future studies, it may be interesting to address the Einstein field equations to understand any further conditions/constraints that may be satisfied by the coefficients of the Bianchi II spacetime metric, in particular, in those cases for which the Noether symmetry classification is obtained.

REFERENCES

[1] Lie, Sophus (1881),"ber die integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen". Arch. for Math (in German) 6: 328368

[2] Lie, Sophus (1890). Theorie der Transformationsgruppen (in German). Teubner, Leipzig.

[3] Lie, Sophus (1893). Theorie der Transformationsgruppen (in German) 3.Teubner,Leipzig.

[4] Cantwell, B. J. (2002). *Introduction to Symmetry Analysis.* New York: Cambridge University Press.

[5] W.Bluman, G.,C.Anco, S. (2002). *Symmetry and Integration Methods of differential Equations.* New York: Springer.

[6] Olver, Peter J.(1993). *Applications of Lie groups to differential equations.*(second ed.). Springer-Verlag.

[7] Stephani, H.(1989) . *Differential Equations* (First ed.). Cambridge University Press.

[8] Bluman, G.; Kumei, S. (1989). *Symmetries and differential equations*. Applied mathematical Science Series 81 (Second ed.). New York: Springer-Verlag.

[9] N H Ibragimon, (1994) *CRC handbook of Lie group analysis of differential equations*, Volume1, Symmetries, exact solutions and conservation laws, CRC Press, Boca Raton.

[10] M. Aguirre and J. Krause, *Infinitesimal symmetry transformations II, Some one-dimensional nonlinear systems*, Journal Mathematical Physics (26), 1985, 593.

[11] W.F Ames and C. Rogers eds. (1992). *Nonlinear equations in applied sciences*, Academic Press, New York.

[12] D Levi, S Tremblay and P Winternitz, *Lie point symmetries of differential equations and lattices*, Journal of Physics A: Mathematical and General (33), doi: 10.1088/0350-4470/33/47/313, (2000).

[13] Bokhari, Ashfaque H., et al. "Wave equation on spherically symmetric Lorentzian metrics." Journal of Mathematical Physics 52 (2011): 063511.

[14] Ahmad, Ahmad M., Ashfaque H. Bokhari, and Fiazud Din Zaman. "Symmetries, Conservation Laws, and Wave Equation on the Milne Metric. "Journal of Applied Mathematics (2012).

[15] Ashfaque H. Bokhari A.H. Kara M. Karim , F.D. Zaman, Invariance Analysis and Variational Conservation Laws for the Wave Equation on Some Manifolds, International journal of Theoretical Physics, 48, (2009) 48: 1919.

[16] Charles W. Misner, John Archibald Wheeler. 'Gravitation' (first ed.), New York (1973).

[17] A. E. Noether, Nachr. Akad. Wiss. Gottingen Math.-Phys. K1 II (1918)

[18] D. D. Djukic, B. Vujanovic, Acta Mech. 23 (1975) 17.

[19] Yi, Zhang, Shang Mei, and Mei Feng-Xiang. "Symmetries and conserved quantities for systems of generalized classical mechanics." Chinese Physics9.6 (2000): 401.

[20] Mei, Shang, Guo Yong-Xin, and Mei Feng-Xiang. "A Birkhoff-Noether method of solving differential equations." Chinese Physics 16.2 (2007): 292.

[21] Capozziello, Salvatore, et al. "Noether symmetries in Bianchi universes." arXiv preprint gr-qc/9606050 (1996).

[22] Ibragimov, N. H., A. H. Kara, and F. M. Mahomed. "Lie-B?cklund and Noether symmetries with applications." Nonlinear Dynamics 15.2 (1998): 115-136.

[23] Camci, Ugur, and Yusuf Kucukakca. "Noether symmetries of Bianchi I, Bianchi III, and Kantowski-Sachs spacetimes in scalar-coupled gravity theories."Physical Review D 76.8 (2007): 084023.

[24] Bokhari, Ashfaque H., and A. H. Kara. "Noether versus Killing symmetry of conformally flat Friedmann metric." General Relativity and Gravitation 39.12 (2007): 2053-2059.

[25] Tsamparlis, Michael, and Andronikos Paliathanasis. "The geometric nature of Lie and Noether symmetries." General Relativity and Gravitation 43.6 (2011): 1861-1881.

[26] Jamil, Mubasher, Fazal M. Mahomed, and D. Momeni. "Noether symmetry approach in f (R)-tachyon model." Physics Letters B 702.5 (2011): 315-319.

[27] Bokhari, A. H., et al. "Symmetry structures and conservation laws of Petrov III and Papapetrou metrics." Indian Journal of Physics (2013): 1-6.

[28] Camci, U., and E. Sahin. "Matter collineation classification of Bianchi type II spacetime." General Relativity and Gravitation 38.9 (2006): 1331-1346.

[29] Dick, Auguste (1981), Emmy Noether: 1882-1935, Boston: Birkhuser. Trans. H. I. Blocher.

[30] Weyl, Hermann (1935), "Emmy Noether", Scripta Mathematica 3 (3): 201-220.

[31] Lederman, Leon M.; Hill, Christopher T (2004), Symmetry and the Beautiful Universe, Amherst: Prometheus Books.

Vitae

- Name: Mahmood Rajih Tarayrah

- Nationality: Palestinian

- Date of Birth: 23/06/1988

- Email: *mtarayrah@yahoo.com*

- Present Address: Dhahran, Saudi Arabia

- Permanent Address: West Bank, Palestine

- At 2010, I received my B.Sc in Mathematics from Hebron University in Hebron, West Bank, Palestine.

- I have two years experience in teaching.

- Now I am under scholarship in Mathematics and Statistics Department at King Fahd University of Petroleum and Minerals to obtain M.Sc. degree in Mathematics.